Witten spinors on nonspin manifolds

Anda Degeratu* and Mark Stern†

Abstract. Motivated by Witten's spinor proof of the positive mass theorem, we analyze asymptotically constant harmonic spinors on complete asymptotically flat nonspin manifolds with nonnegative scalar curvature.

1. Introduction

Techniques from spin geometry have proved very powerful in the study of positive scalar curvature. However, not all manifolds are spin. The topological obstruction to the existence of a spin structure is the second Stiefel-Whitney class. In this article we explore, in the context of the positive mass theorem, whether it is possible to graft some of the machinery of spin geometry onto the study of nonspin manifolds.

The fundamental idea is to excise a representative of the Poincaré dual of the second Stiefel-Whitney class and apply spin techniques on the complement. We explore this idea by trying to adapt Witten's proof of the positive mass theorem to the nonspin case. The difficulties in executing this approach arise from the fact that this complement is incomplete, greatly complicating analytic arguments. Ultimately, we do not succeed in this endeavor, but we hope that our analysis of harmonic spinors on incomplete spin manifolds may prove useful in other contexts. A prior examination of incomplete spin structures appears in [Bal].

1.1. The positive mass theorem

Definition 1.1. A complete non-compact Riemannian manifold (M^n, g) is called *asymptotically flat* of order $\tau > 0$ if there exists a compact set, $K \subset M$, whose complement is a disjoint union of subsets $M_1, \ldots, M_L$ – called the *ends* of M – such that for each end there exists a diffeomorphism

$$Y_l : \mathbb{R}^n \setminus B_T(0) \to M_l,$$

so that $Y_l^* g =: g_{ij} dx^i dx^j$ satisfies for $\rho = |x|$,

$$g_{ij} = \delta_{ij} + \mathcal{O}(\rho^{-\tau}), \quad \partial_k g_{ij} = \mathcal{O}(\rho^{-\tau-1}), \quad \partial_k \partial_l g_{ij} = \mathcal{O}(\rho^{-\tau-2})$$

with δ the Euclidean metric on $\mathbb{R}^n$, and $T > 1$. We call such a coordinate chart (M_l, Y_l) *asymptotically flat.*

*University of Freiburg, Mathematics Institute; email: anda.degeratu@math.uni-freiburg.de
†Duke University, Department of Mathematics; email: stern@math.duke.edu

For an end of an asymptotically flat manifold, the *mass* is defined to be

$$\mathrm{mass}(M_l, g) := \frac{1}{c(n)} \lim_{\rho\to\infty} \int_{S_\rho^{n-1}} \frac{x^j}{\rho}(\partial_i g_{ij} - \partial_j g_{ii}) d\sigma, \tag{1.2}$$

if the limit exists. Here $c(n)$ is a normalizing constant depending only on the dimension of the manifold, and S_ρ^{n-1} is the sphere of radius ρ in the asymptotically flat coordinate chart (M_l, Y_l). The *mass* of the manifold (M^n, g) is the sum of the masses of its ends

$$\mathrm{mass}(M, g) := \sum_{l=1}^{L} m(M_l, g).$$

Bartnik showed that if (M, g) satisfies the *mass decay conditions*,

$$\tau > \frac{n-2}{2} \qquad \text{and} \qquad R \in L^1(M), \tag{1.3}$$

then the mass is well-defined and is a Riemannian invariant, [Bar]. Here R denotes the scalar curvature of g.

Positive Mass Theorem. *Let (M, g) be an asymptotically flat Riemannian manifold of dimension $n \geq 3$ satisfying the mass decay conditions* (1.3) *with nonnegative scalar curvature. Then the mass is nonnegative. Futhermore* $\mathrm{mass}(M, g) = 0$ *if and only if (M, g) is isometric to the Euclidean space.*

The positive mass theorem has a long history. Arnowitt, Deser and Misner introduced the notion of mass of an asymptotically flat spacelike hypersurface in space-time and conjectured its positivity for 3-dimensional spacelike hypersurfaces. This conjecture was proved by Schoen and Yau, using minimal surface techniques. Their proof extends readily to dimensions $n \leq 7$. Witten subsequently gave a different argument using spinors, which proved the conjecture for *spin* manifolds of arbitrary dimension. The requisite analysis was provided by Parker and Taubes [PT] and Bartnik [Bar]. After this work, the positive mass theorem was open for higher dimensional *nonspin* manifolds.

Recently, Schoen [Sc1] and Lohkamp [Lo] have each announced programs for extending to higher dimension the minimal surface approach to proving the positive mass theorem. On the other hand, our analysis of harmonic spinors on nonspin manifolds is motivated by Witten's proof, which we now briefly recall.

1.2. Witten's proof of the positive mass theorem

Let (M, g) be an asymptotically flat Riemannian spin manifold of dimension n which satisfies the mass decay conditions (1.3). For simplicity, we assume here that M has only one end. Let ψ_0 be a smooth spinor on M which is constant near infinity with respect to an asymptotically flat coordinate system and normalized by $|\psi_0|^2 \to 1$ at infinity. Then there exists a unique spinor u, with $Du \in L^2(M, S)$ and $\frac{u}{\rho} \in L^2(M, S)$ solving

$$D^2 u = -D\psi_0. \tag{1.4}$$

Let

$$\psi := Du + \psi_0. \tag{1.5}$$

Applying the Lichnerowicz formula to the harmonic spinor ψ one computes (see the proof of Proposition 4.15) that

$$\int_M |\nabla\psi|^2 + \frac{R}{4}|\psi|^2 - |D\psi|^2 = \frac{c(n)}{4}\,\mathrm{mass}(M,g). \tag{1.6}$$

Since the spinor ψ is harmonic and the scalar curvature R is positive and L^1, the mass of (M,g) is finite and nonnegative.

1.3. Our main results

In this work we study a natural extension of Witten's argument to nonspin manifolds. Suppose that (M,g) is a nonspin Riemannian manifold that is asymptotically flat of order $\tau > 0$. By passing to an oriented double cover when necessary, it sufficies to consider the case of orientable manifolds. In Theorem 2.7 we show that we can choose a compact subset V in the compact part K of M, stratified by smooth submanifolds V^{k_b} of codimension $k_b = b(b-1)$ with $b \geq 2$, so that $M \setminus V$ admits a spin structure, and this spin structure does not extend over the lowest codimension stratum V^2. We fix such a V and the corresponding spin structure on $M \setminus V$. This spin structure restricts to the trivial spin structure on each of the asymptotically flat ends of M. We denote with S the corresponding spinor bundle on $M \setminus V$.

To implement Witten's proof, we first need to construct an asymptotically constant harmonic spinor, which we call a "Witten spinor". The properties of this spinor are exactly those properties of the spinor ψ constructed above in Witten's proof on a complete spin manifold.

Definition 1.7. Let ψ_0 be a spinor on $M \setminus V$, constant near infinity. We say that a spinor ψ on $M \setminus V$ is a *Witten spinor* asymptotic to ψ_0, if the following conditions are satisfied :

1. $\frac{\psi-\psi_0}{\rho} \in L^2(M \setminus V, S)$,
2. ψ is strongly harmonic, i.e. $D\psi = 0$, and
3. $\nabla(\psi - \psi_0) \in L^2(M_l, S|_{M_l})$ for each asymptotically flat end M_l of M.

We show that such spinors exist on $M \setminus V$.

Theorem A. *Let (M,g) be a nonspin Riemannian manifold which is asymptotically flat of order $\tau > \frac{n-2}{2}$ and which has nonnegative scalar curvature. Given a smooth spinor ψ_0 on $M \setminus V$ that is constant near infinity and that vanishes in a neighborhood of V, there exists a Witten spinor on $M \setminus V$ asymptotic to ψ_0.*

Next one needs to use the integral form of the Lichnerowicz formula on the *incomplete* manifold $M \setminus V$. In this case, integration by parts may introduce unwanted boundary terms from the ideal boundary V into the formula. This is a problem familiar in the study of the Hodge theory of L^2-cohomology of singular varieties (see [PS2]), which is resolved in that context by proving that, as they approach the singularities, harmonic forms decay sufficiently rapidly to introduce no extra boundary terms when integrating by parts.

To analyze the behaviour of the Witten spinors near V, we study the growth of ψ near each of the strata V^{k_b} of V separately. Unless V^{k_b} is a closed stratum, there are no tubular neighborhoods of uniform radius over the entire stratum. Hence, for uniform estimates involving separation of variables, we formulate our estimates in tubular neighborhoods over relatively compact subsets of V^{k_b} that do not intersect the higher codimension strata. We denote by $\mathrm{TRC}(V^{k_b})$ the set of all these good neighborhoods around points in V^{k_b}. Letting r denote the distance to V^2 the lowest codimension stratum, and r_b denote the distance to the higher codimension strata V^{k_b} of V, we have:

Theorem B. *Let ψ be a Witten spinor constructed as in Theorem A. Then*

1. *for all $W \in \mathrm{TRC}(V^2)$*

$$\frac{\psi}{r^{1/2}\ln^{1/2+a}(\frac{1}{r})} \in L^2(W \setminus V, S|_{W\setminus V}), \quad \textit{for all } a > 0, \tag{1.8}$$

2. *for all $W \in \mathrm{TRC}(V^{k_b})$ with $k_b > 2$*

$$\frac{\psi}{r_b^{(k_b-2)/2}\ln^{1/2+a}(\frac{1}{r_b})} \in L^2(W \setminus V, S|_{W\setminus V}), \quad \textit{for all } a > 0. \tag{1.9}$$

However, the decay estimates in (1.8) are borderline for our purposes. For any class of manifolds for which we could set $a = 0$ in (1.8), Witten's proof of the positive mass theorem extends.

Theorem C. *Let (M, g) be an asymptotically flat nonspin manifold which satisfies the hypothesis of the Positive Mass Theorem. If the Witten spinor constructed in Theorem A has the property*

$$\frac{\psi}{r^{1/2}\ln^{1/2}(\frac{1}{r})} \in L^2(W \setminus V, S|_{W\setminus V}) \tag{1.10}$$

for all $W \in \mathrm{TRC}(V^2)$, then the mass of (M, g) is nonnegative.

Since the spin structure on $M \setminus V$ does not extend over V^2, spinors have nontrivial holonomy around small circles normal to V^2. The L^2-harmonic spinors near V^2 have a Fourier decomposition in these normal circles whose leading order modes in polar coordinates may behave like $r^{-1/2}e^{\pm\frac{i\theta}{2}}$. Such modes prevent direct application of the Lichnerowicz formula. But, if a spinor satisfies the hypotheses of Theorem C, these modes vanish, giving that its product with any element of $C_0^\infty(M)$ is in the minimal domain of the Dirac operator. On the other hand, the decay obtained in Theorem B near V^2 is not sufficient to remove them. Hence, our estimates do not suffice to extend Witten's proof of the positive mass theorem.

1.4. Plan of the Paper

In Section 2, starting from the fact that the second Stiefel-Whitney class vanishes on any bundle admitting a rank $(n-1)$ trivial subbundle, we construct the stratified space V whose complement is an incomplete spin manifold. We then analyze the geometric structure of V and introduce the set $\mathrm{TRC}(V^{k_b})$ of good tubular neighborhoods.

In Section 3 we gather Poincaré estimates for spinors on the asymptotically flat ends of M and near each stratum of V. Near the codimension 2 stratum of V we have a stronger *angular estimate*. This is a consequence of the fact that the spin structure on $M \setminus V$ does not extend over V^2, and the resulting nontrivial holonomy around small normal circles to V^2.

We follow with a preliminary study of the Dirac operator D on $M \setminus V$ in Section 4. We introduce the weighted Sobolev spaces necessary for our analysis, state the Lichnerowicz formula on these spaces, and derive a vanishing result in Proposition 4.6. We also introduce the maximal and the minimal domains of D. We continue with an analysis of the growth condition of the Witten spinors on the asymptotically flat ends, and conclude this section with Proposition 4.15, which recapitulates Bartnik's proof that the asymptotic boundary contribution to the Lichnerowicz formula for strongly harmonic spinors does yield the product of a universal constant and the mass.

In Section 5 we derive estimates near each stratum of V for spinors u and $v := Du$, with u in the minimal domain of D. In Lemma 5.7, we also derive a sufficient condition for a spinor to be in the minimal domain of D. In Section 6 we sharpen these estimates when $D^2 u = 0$ near V. Both these sections rely on the estimates established in Section 3 and a technique of Agmon which we use repeatedly, [Ag].

In Section 7 we derive estimates for spinors on the asymptotically flat ends of M.

In Section 8 we prove a coercivity result for the Dirac operator, which we then use in Section 9 together with Corollary 4.14 to prove Theorem A. We conclude this section by proving Theorems B and C.

Acknowledgements: We would like to thank Hubert Bray for helpful discussions. The first author would like to thank Tom Mrowka and Richard Melrose for support and useful discussions, and also acknowledge the support of NSF DMS-0505767 and the Max Planck Institute for Gravitational Physics in Potsdam, Germany during the time this work was carried out. The work of the second author was supported partially by NSF grant DMS-1005761.

Contents

2. Incomplete Spin Structures

Let M be an oriented n-dimensional Riemannian manifold which is nonspin. Thus it has non-vanishing second Stiefel-Whitney class, $w_2(M)$. Based on the definition of $w_2(M)$, we construct a compact subset $V \subset M$ so that $M \setminus V$ admits a spin structure.

2.1. The construction of V

The second Stiefel-Whitney class is the topological obstruction to extending $(n-1)$ linearly independent vector fields from the 1-skeleton to the 2-skeleton of M. In particular, it vanishes for any manifold admitting $(n-1)$ pointwise linearly independent vector fields. With this in mind, we consider the vector bundle $\mathrm{Hom}(\mathbb{R}^{n-1}, TM)$. The fiber at each point x is identified with the space of $(n-1)$-tuples of vectors in T_xM. In this fiber we take the subset $\mathcal{H}_x$ of maps which are not of maximal rank. Set $\mathcal{H} := \cup_{x\in M}\mathcal{H}_x$. This is a stratified set, with strata occurring only in codimension $k_b := b(b-1)$ with $b \geq 2$. The codimension k_b stratum, $\mathcal{H}^{k_b}$, consists of all the points (x,T) with T of rank $n-b$, $b \geq 2$. The codimension is exactly the dimension of the space of normal deformations $\mathcal{N}_{(x,T)} = \mathrm{Hom}(\mathrm{Ker}\, T, \mathrm{Coker}\, T)$ at $(x,T) \in \mathcal{H}$.

We choose a section s of $\mathrm{Hom}(\mathbb{R}^{n-1}, TM)$, transverse to $\mathcal{H}$; it corresponds to an $(n-1)$-tuple of vector fields on M. Let $\Sigma = \Sigma(s)$ be the set where these vector fields fail to be linearly independent. It is a stratified space, with strata Σ^{k_b} only in codimension k_b :

$$\Sigma = \bigcup_{b\geq 2} \Sigma^{k_b} \; = \; \Sigma^2 \cup \Sigma^6 \cup \Sigma^{12} \cup \dots .$$

By construction, $M \setminus \Sigma$ admits a spin structure. Let $P_{\mathrm{Spin}}(M \setminus \Sigma)$ be the associated lifting of the bundle of orthonormal frames, $P_{\mathrm{SO}}(M \setminus \Sigma)$. However, it might be possible to extend this spin structure over certain connected components of Σ^2. In the next Lemma we do exactly this: we extend the spin structure as much as possible over Σ^2.

Lemma 2.1. *There exists $V \subset \Sigma$, a closed stratified space*

$$V = \bigcup_{b\geq 2} V^{k_b} \; = \; V^2 \cup V^6 \cup V^{12} \cup \dots \tag{2.2}$$

such that $M \setminus V$ is spin, but the spin structure $P_{\mathrm{Spin}}(M \setminus \Sigma)$ cannot be extended over $\Sigma^2 \setminus V^2$.

Proof. Let $x \in \Sigma^2$. The holonomy of $P_{\mathrm{Spin}}(M \setminus \Sigma)$ on infinitesimally small loops in the transverse slice to Σ at x is either $+1$ or -1. If the holonomy is $+1$ then it is so for the entire connected component of Σ^2 in which x lies. The spin structure extends over this connected component.

We take V^2 to consist of those connected components of Σ^2 around which the holonomy of an infinitesimal loop is -1; and then define $V^{k_b} := \Sigma^{k_b}$ for the higher codimension strata. The fact that V is closed follows since Σ and $\mathcal{H}$ are closed, as locally $\mathcal{H}$ is given by the vanishing of the determinants of $(n-1) \times (n-1)$ minors of a $(n-1) \times n$ matrix. $\square$

This Lemma allows us to choose a spin structure over $M \setminus V$ with the property that it does not extend over any component of V^2. We call such a spin structure *maximal*.

Moreover, since M, as an asymptotically flat manifold, has a natural spin structure on the asymptotically flat end, the set V can be chosen to lie in the compact part of M.

Lemma 2.3. *Let M be an asymptotically flat manifold which is not spin. Then the stratified space V can be chosen so that it lies in the compact part K of M. The resulting spin bundle is trivial on each end.*

Proof. We show that we can choose a generic section s of the bundle $\mathrm{Hom}(\mathbb{R}^{n-1}, TM)$ so that $\Sigma(s) \subset K$ and closed. Choose asymptotically flat coordinates x on each end. Choose the section $s := \langle \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n-1}} \rangle$, and then extend s to a section of $\mathrm{Hom}(\mathbb{R}^{n-1}, TM)$, transverse to $\mathcal{H}$. By construction, $\Sigma(s) \subset K$.

The distinct spin structures on each end are parameterized by $H^1(\mathbb{R}^n \setminus B_T(0), \mathbb{Z}_2) = 0$, for $n > 2$. Hence the spin bundle must agree with the trivial spin bundle on each end. $\square$

2.2. A remark about higher codimension

We saw in Lemma 2.1 that the spin structure can be extended over those connected components of Σ^2 where the holonomy of $P_{\mathrm{Spin}}(M \setminus \Sigma)$ is $+1$. We show now that the spin structure can be extended over any of the higher codimension components of Σ^{k_b} with $b > 2$ which is a smooth closed submanifold. A similar result was obtained in the PhD thesis of Baldwin [Bal, Section 2.1.2].

Presumably, it should also be possible to extend the spin structure over any of the connected components of V that do not intersect V^2. In particular, if $V^2 = \emptyset$ then we expect that M is spin. If this is not, in fact the case, then our construction of Witten spinors and our attendant regularity results show that the positive mass theorem holds for such nonspin manifolds with 'small' singular set V.

Proposition 2.4. *Let X be an n-dimensional smooth orientable manifold. Let Y be a smooth connected closed submanifold of codimension k. Assume that $X \setminus Y$ is spin. If $k \geq 3$, then X is spin.*

Proof. Choose a Riemannian metric on X, and let $p : D(Y) \to Y$ be the normal disk bundle to Y and $S(Y)$ the corresponding sphere bundle. We identify $D(Y)$ with a tubular neighborhood of Y with boundary $S(Y)$. Let $w_k(N)$ denote the kth Stiefel-Whitney class of the normal bundle NY to Y in X. Consider the Gysin sequence with mod 2 coefficients for the unoriented sphere bundle $S(Y)$ (see [MS, p.144]):

$$\cdots \longrightarrow H^{i-k}(Y, \mathbb{Z}_2) \xrightarrow{\cup w_k(N)} H^i(Y, \mathbb{Z}_2) \xrightarrow{p^*} H^i(S(Y), \mathbb{S}_2) \longrightarrow H^{i+1-k}(Y, \mathbb{Z}_2) \longrightarrow \cdots. \tag{2.5}$$

Since $k \geq 3$, we see that for $i = 2$, the map $p^* : H^2(Y, \mathbb{Z}_2) \to H^2(S(Y), \mathbb{Z}_2)$ is an injection, while for $i = 1$, $p^* : H^1(Y, \mathbb{Z}_2) \to H^1(S(Y), \mathbb{Z}_2)$ is an isomorphism.

Next, we consider the Mayer-Vietoris sequence

$$\cdots \longrightarrow H^1(X,\mathbb{Z}_2) \longrightarrow H^1(D(Y),\mathbb{Z}_2)\oplus H^1(X\setminus Y,\mathbb{Z}_2) \longrightarrow H^1(S(Y),\mathbb{Z}_2) \xrightarrow{\delta}$$

$$\xrightarrow{\delta} H^2(X,\mathbb{Z}_2) \longrightarrow H^1(D(Y),\mathbb{Z}_2)\oplus H^1(X\setminus Y,\mathbb{Z}_2) \longrightarrow H^1(S(Y),\mathbb{Z}_2) \longrightarrow \cdots$$

In this sequence, upon identifying $H^i(D(Y),\mathbb{Z}_2) \equiv H^i(Y,\mathbb{Z}_2)$, the map $H^i(D(Y),\mathbb{Z}_2) \to H^i(S(Y),\mathbb{Z}_2)$ is the same as the map p^* in the Gysin sequence (2.5). Hence the map $H^1(D(Y),\mathbb{Z}_2) \to H^1(S(Y),\mathbb{Z}_2)$ is surjective, and the map $H^2(D(Y),\mathbb{Z}_2) \to H^2(S(Y),\mathbb{Z}_2)$ is injective. This also implies that the boundary map δ is identically zero.

By the naturality of the Stiefel-Whitney classes, the image of $w_2(X)$ in $H^2(X\setminus Y,\mathbb{Z}_2)$ is $w_2(TX|_{X\setminus Y})$. This vanishes since $X\setminus Y$ is spin. On the other hand, the image of $w_2(X)$ in $H^2(D(Y),\mathbb{Z}_2)$ is zero by the aformentioned injectivity and the fact that $TX|_{S(Y)}$ is spin, as $S(Y)$ is the boundary of the spin manifold $X\setminus D(Y)$. Hence $w_2(X)=0$, and thus X is spin. ☐

Remark 2.6. Note that in codimension $k=2$ this proof breaks at two stages: First of all, for $i=1$ in the Gysin sequence (2.5), the map $p^*: H^1(Y,\mathbb{Z}_2) \to H^1(S(Y),\mathbb{Z}_2)$ is not necessarily surjective, and thus we cannot conclude that $\delta=0$ in the Mayer-Vietoris sequence. Secondly, since for $i=2$ the map $p^*: H^2(Y,\mathbb{Z}_2) \to H^2(S(Y),\mathbb{Z}_2)$ is not injective, we cannot conclude that the image of $w_2(X)$ in $H^2(D(Y),\mathbb{Z}_2)$ vanishes.

2.3. The conical structure of the singularities of V

The singular structure of the set V defined in Lemma 2.1 is easily deduced from the geometry of the subset $\mathcal{H}$ of $\mathrm{Hom}(\mathbb{R}^{n-1},TM)$. Each stratum $\mathcal{H}^{k_b}$ lies in the closure of each of the higher dimensional strata (the strata $\mathcal{H}^{k_a}$ with $a<b$). For $(x,T)\in\mathcal{H}^{k_b}$, the normal bundle to this stratum within $\mathrm{Hom}(\mathbb{R}^{n-1},TM)$ can be identified with the stratified space of maps in $\mathrm{Hom}(\mathrm{Ker}\,T,\mathrm{Coker}\,T)$. The elements which are not of maximal rank can be identified with a subcone of the normal bundle. Elements in this subcone exponentiate to $\mathcal{H}$. A choice of coordinates in a neighborhood U of x allows us to locally trivialize these structures and to identify, via the exponential map, a neighborhood of (x,T) in $\mathcal{H}$ with the product of a neighborhood of (x,T) in $\mathcal{H}^{k_b}$ and a small cone of nonmaximal rank elements in $\mathrm{Hom}(\mathbb{R}^{b-1},\mathbb{R}^b)$. The cone can be realized as a cone over a subvariety of nonmaximal rank elements in a small sphere in $\mathrm{Hom}(\mathbb{R}^{b-1},\mathbb{R}^b)$. This subvariety is again a stratified space. Therefore, the point (x,T) has a neighborhood which is a product of a manifold and a cone over the stratified space $S^{k_b-1}\cap\mathcal{H}$. Inducting both on the dimension of the manifold and the dimension of the strata, we see that $\mathcal{H}$ is locally quasi-isometric to an iterated cone. This is the familiar cone over cone topological structure of singularities of projective varieties arising here as a geometric structure. This geometric cone structure is preserved under pull-back by transversal maps, and thus inherited by V.

We collect the discussion so far in the following theorem.

Theorem 2.7. *Let (M,g) be an asymptotically flat Riemannian manifold which is nonspin. Then there exists a closed stratified subset V, locally quasi-isometric to an iterated cone and*

lying in the compact part K of M, so that the spin structure on $M \setminus V$ is maximal, in the sense that it does not extend over any of the codimension 2 strata of V. The strata of V,

$$V = V^{k_2} \cup V^{k_3} \cup \dots V^{k_{d-1}} \cup V^{k_d},$$

have codimensions $k_b = b(b-1)$ in M. Moreover, the maximal spin structure on $M \setminus V$ is trivial on each assymptotically flat end of M.

For the reminder of the paper we fix such a V and the associated maximal spin structure. We denote by S the corresponding spinor bundle.

2.4. The geometric structure near V

To simplify our analysis near V, we introduce a special set of tubular neighborhoods adapted to the geometry of the stratified set V.

Remark 2.8. Here we assume that we are in the topologically generic situation where all V^{k_b} are nonempty, $2 \le b \le d$. For general M, some of the strata of $\mathcal{H}$ might give empty strata of V. This affects the following discussion only notationally.

We employ the convention that all tubular neighborhoods are geodesic, have constant radius, and do not intersect the higher codimension strata of V. Unless V^{k_b} is a closed stratum, there are no tubular neighborhoods over the whole stratum. Hence, we must restrict attention to tubular neighborhoods over relatively compact subsets Y of V^{k_b}. The larger we take Y, the smaller we must take the radius of the tube.

Consider V^{k_b}, one of the strata of V. Let T^{k_b} be a rotationally symmetric neighborhood of the zero section of the normal bundle N^{k_b} of V^{k_b} in M on which the exponential map is a diffeomorphism. Let $\mathcal{W}^{k_b} := \exp(T^{k_b})$. On $\mathcal{W}^{k_b}$ we define a normal distance function r_b, by setting

$$r_b(\exp_x(v)) := |v| \tag{2.9}$$

for each $x \in V^{k_b}$ and $v \in N^{k_b}_x \cap T^{k_b}$.

Given any relatively compact subset $Y \subset V^{k_b}$, there exists $0 < \epsilon(Y) < \frac{1}{2}$ so that the tubular neighborhood of any radius $\epsilon < \epsilon(Y)$ over Y is contained in $\mathcal{W}^{k_b}$ and does not intersect $\overline{V^{k_{b+1}}}$. Denote this neighborhood $B_\epsilon(Y)$.

Set

$$\mathrm{TRC}(V^{k_b}) := \{B_\epsilon(Y) \subset \mathcal{W}^{k_b} : Y \text{ is a relatively compact subset of } V^{k_b} \text{ and } \epsilon < \epsilon(Y)\}. \tag{2.10}$$

Note that we have $W \cap \overline{V^{k_{b+1}}} = \emptyset$ for all $W \in \mathrm{TRC}(V^{k_b})$.

Remark 2.11. On each $W \in \mathrm{TRC}(V^{k_b})$ we have a well-defined normal distance function r_b. Our choice $\epsilon(Y) < \frac{1}{2}$ implies that r_b is always less than $\frac{1}{2}$.

We now study the metric on these tubular neighborhoods. The discussion is similar to the discussion in [Gr, Section 2]. Consider $W = B_\epsilon(Y)$, with (Y, ϕ) a coordinate neighborhood in V^{k_b} with coordinate functions $\phi(y) = (y^1, \dots y^{n-k_b})$. Also choose a trivialization of the

normal bundle N^{k_b} on Y, with $\{n_\alpha\}_{\alpha=1}^{k_b}$ an orthonormal frame on $N^{k_b}|_Y$. Using this, we have the following coordinates (y^i, t^α) on W:

$$\Psi : \phi(Y) \times B_\epsilon(0) \subset \mathbb{R}^{n-k_b} \times \mathbb{R}^{k_b} \to W \subset M,$$

with

$$\Psi(y^i, t^\alpha) = \exp_y\Big(\sum_{\alpha=1}^{k_b} t^\alpha\, n_\alpha(y)\Big). \tag{2.12}$$

Using these coordinates, we write the metric near Y as

$$g = g_{ij} dy^i dy^j + g_{\alpha\beta} dt^\alpha dt^\beta + g_{i\alpha} dy^i \circ dt^\alpha,$$

where $g_{i\alpha}$ vanishes on $Y \subset V^{k_b}$ and $g_{\alpha\beta} - \delta_{\alpha\beta} = \mathcal{O}(|t|^2)$. Moreover, the distance function r_b defined in (2.13) is $r_b = |t|$. With this, we define the *radial vector* of this tubular neighborhood

$$\frac{\partial}{\partial r_b} := \sum_{\alpha=1}^{k_b} \frac{t^\alpha}{|t|} \frac{\partial}{\partial t^\alpha}. \tag{2.13}$$

It satisfies $\nabla_{\frac{\partial}{\partial r_b}} \frac{\partial}{\partial r_b} = 0$, and

$$|dr_b|^2 = 1. \tag{2.14}$$

It also follows that

$$\frac{\partial}{\partial r_b}\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial r_b}\rangle = \langle \nabla_{\frac{\partial}{\partial r_b}} \frac{\partial}{\partial y^i}, \frac{\partial}{\partial r_b}\rangle = -\langle \nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial r_b}, \frac{\partial}{\partial r_b}\rangle = 0,$$

and hence

$$\langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial r_b}\rangle = 0,$$

and

$$\sum_\alpha t^\alpha g_{i\alpha} = 0.$$

2.5. Behaviour of the spin structure near the codimension 2 stratum of V

Let $W \in \mathrm{TRC}(V^2)$. From (2.10), it follows that there exists $Y \subset V^2$ relatively compact so that $W = B_\epsilon(Y)$ for some $\epsilon < \epsilon(Y)$, and that W does not intersect $\overline{V^6}$, the higher codimension strata of V. We denote by $r := r_2$, the normal distance function to V^2. Let (r, θ) be the polar coordinates in each normal disk $D_y \subset W$ to $y \in Y$.

We choose W to be contractible. It admits a trivial spin structure, with spinor bundle S_0. The two spinor bundles $S|_{W\setminus Y}$ and S_0 become trivial when lifted to a connected double cover of $W \setminus Y$. We use this lifting to identify sections of S with multivalued sections of S_0. In particular, the maximality of the spin structure near V implies the following lemma.

Lemma 2.15. *Let $W = B_\epsilon(Y) \in \mathrm{TRC}(V^2)$ contractible. Then with respect the above identification on the double cover of $W \setminus Y$, each spinor ψ on $W \setminus Y$ satisfies*

$$\psi(y, r, \theta + 2\pi) \;=\; -\psi(y, r, \theta), \tag{2.16}$$

on each disk $D_y \subset W$, the normal disk to $y \in Y$. Therefore the Fourier decomposition of ψ in this transverse disk is

$$\psi(y, r, \theta) = \sum_{k \in \frac{1}{2} + \mathbb{Z}} \psi^k(y, r)\, e^{ik\theta}. \tag{2.17}$$

Proof. The first claim follows from the fact that the holonomy of the spin connection is -1.

Let now $\{f^a\}$ be a frame giving a trivialization of the trivial spinor bundle S_0 over W. In this frame

$$\psi(y,r,\theta)=\sum_a \psi_a(y,r,\theta)f^a=\sum_a \sum_{k\in\frac{1}{2}+\mathbb{Z}} \psi_a^k(y,r)e^{ik\theta}f^a,$$

with $\psi_a^k(y,r):=\frac{1}{2\pi}\int_0^{2\pi}\psi_a(y,r,s)e^{-iks}ds$. Defining

$$\psi^k(y,r):=\sum_a \psi_a^k(y,r)f^a$$

we obtain (2.17). $\square$

The individual Fourier components $\psi^k(x,r)$ in the above expansion are dependent on the choice of the trivialization of S_0 and so are not globally defined. The next lemma gives the dependance of this expansion on the trivialization

Lemma 2.18. *Let $W=B_\epsilon(Y)\in \mathrm{TRC}(V^2)$ contractible. Let ψ be a spinor on $W\setminus Y$, and let $\psi^k(x,r)$ and $\hat\psi^k(x,r)$ be the Fourier components of ψ with respect to two different trivializations of S_0. Then*

$$\|\psi^k-\hat\psi^k\|_{L^2(W\setminus Y,S|_{W\setminus Y})}\le C\|r\psi\|_{L^2(W\setminus Y,S|_{W\setminus Y})}.$$

Here the constant $C>0$ is independent of ψ but does depend essentially on the two trivializations.

Proof. We continue with the set-up in the proof of Lemma 2.15. Consider another frame $\{h^b\}$ giving a trivialization of the spinor bundle S_0 over W. In this other trivialization, we have

$$\psi(y,r,\theta)=\sum_{k\in\frac{1}{2}+\mathbb{Z}}\hat\psi^k(y,r)e^{ik\theta},$$

with

$$\hat\psi^k(y,r):=\sum_b \hat\psi_b^k(y,r)h^b=\frac{1}{2\pi}\sum_b\left(\int_0^{2\pi}\hat\psi_b(y,r,s)e^{-iks}ds\right)h^b$$

There exists a matrix valued map q on W, so that the two frames $\{f^a\}$ and $\{h^b\}$ are related via $h^b=\sum_a q_a^b f^a$. With this, the above becomes

$$\begin{aligned}
\hat\psi^k(y,r)&=\frac{1}{2\pi}\sum_{a,b}\left(\int_0^{2\pi}\hat\psi_b(y,r,s)\ e^{-iks}ds\right)q_a^b(y,r,\theta)f^a\\
&=\frac{1}{2\pi}\sum_{a,b,c}\left(\int_0^{2\pi}\psi_c(y,r,s)(q^{-1}(y,r,s))_b^c\ e^{-iks}ds\right)q_a^b(y,r,\theta)f^a\\
&=\frac{1}{2\pi}\sum_a\left(\int_0^{2\pi}\psi_a(y,r,s)e^{-iks}ds\right)f^a\\
&\quad+\frac{1}{2\pi}\sum_{a,b,c}\left(\int_0^{2\pi}\psi_c(y,r,s)\left((q^{-1}(y,r,s))_b^c q(y,r,\theta)_a^b-\delta_a^c\right)\ e^{-iks}ds\right)f^a.
\end{aligned}$$

Thus

$$\hat{\psi}_k(y,r)-\psi_k(y,r) = \frac{1}{2\pi}\sum_{a,b,c}\left(\int_0^{2\pi}\psi_c(y,r,s)(q^{-1}(y,r,s)-q^{-1}(y,0))_b^c\; q(y,r,\theta)_a^b e^{-iks}ds\right)f^a$$
$$+\frac{1}{2\pi}\sum_{a,b,c}\left(\int_0^{2\pi}\psi_c(y,r,s)q^{-1}(y,0)_b^c\;(q(y,r,\theta)-q(y,0))_a^b e^{-iks}ds\right)f^a$$

Using the Taylor expansion for the components of $q(y,r,\theta)$ and $q^{-1}(y,r,\theta)$, it follows that both terms on the right-hand-side are of the form $\mathcal{O}(r\int_0^{2\pi}|\psi(y,r,\theta)|d\theta)$. We have

$$\|\psi^k(y,r)-\hat{\psi}^k(y,r)\|^2_{L^2(W\setminus Y,S|_{W\setminus Y})} = \int_0^{\epsilon}\int_0^{2\pi}\int_Y|\psi^k(y,r)-\hat{\psi}^k(y,r)|^2 m(y,r,\theta)\; rdrd\theta dy$$

with m smooth and bounded from above and below by strictly positive constants. (Such an m exists since on this neighborhood the metric is quasi-isometric with the product metric on W.) Hence

$$\begin{aligned}
\|\psi^k(y,r)-\hat{\psi}^k(y,r)\|^2_{L^2(W\setminus Y,S|_{W\setminus Y})} &\le C_1\int_0^{\epsilon}\int_0^{2\pi}\int_Y\left(r\int_0^{2\pi}|\psi(y,r,s)|ds)\right)^2 m(y,r,\theta)\; rdrd\theta dy\\
&\le 2\pi C_1\int_0^{\epsilon}\int_0^{2\pi}\int_Y(\int_0^{2\pi}(r|\psi(y,r,s)|)^2ds)m(y,r,\theta)\; rdrd\theta dy\\
&\le 4\pi^2 C_2\int_0^{\epsilon}\int_Y\int_0^{2\pi}(r|\psi(y,r,s)|)^2 m(y,r,s)\; rdrdsdy\\
&= C\|r\psi\|^2_{L(W\setminus Y,S|_{W\setminus Y})}.
\end{aligned}$$

Here C_1 is a positive constant which depends on the two chosen trivializations of S_0, while C_2 is a positive constant depending on the upper and lower bounds of m and on C_1. □

3. Preliminary estimates

We will need estimates for the decay of various spinors on the asymptotically flat end of M and also near the strata of V. We use two types of estimates: radial Poincaré-type estimates, and angular estimates near the codimension 2 stratum of V. The angular estimates are a direct consequence of the fact that the spin structure on $M\setminus V$ has holonomy around small circles normal to V^2.

3.1. Poincaré estimates

In this section we gather some Poincaré inequalities, which we require to control both large scale and small scale behavior of functions and spinors. All the estimates arise as simple perturbations of basic Euclidean Poincaré inequalities for functions, whose proofs we first recall. The passage from inequalities for functions to inequalities for spinors follows from Kato's inequality. All the norms in this section are L^2-norms.

3.1.1. Euclidean Poincaré Inequalities

Proposition 3.1. *Let $f \in H^1(\mathbb{R}^n)$, $n > 2$. Let ρ denote the radial distance. Then*

$$\|df\|^2 \geq \frac{(n-2)^2}{4} \, \|\frac{f}{\rho}\|^2. \tag{3.2}$$

When $n = 2$ we have

$$\|df\|^2 \geq \frac{1}{4} \, \|\frac{f}{\rho \ln(\frac{1}{\rho})}\|^2. \tag{3.3}$$

Proof. The subspace $\mathcal{C}_0^\infty(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$. Hence it suffices to prove the estimate for this subspace. Let $f \in C_0^\infty(\mathbb{R}^n)$, and compute in spherical coordinates:

$$\begin{aligned}
\|\frac{f}{\rho}\|^2 &= \frac{1}{n-2} \int_0^\infty \int_{S^{n-1}} |f|^2 \; d\rho^{n-2} \, d\sigma \\
&= \frac{-2}{n-2} \int_0^\infty \int_{S^{n-1}} f \, \frac{\partial f}{\partial \rho} \; \rho^{n-2} d\rho \, d\sigma \\
&\leq \frac{2}{n-2} \|\frac{f}{\rho}\| \, \|df\|.
\end{aligned}$$

Dividing through by $\|\frac{f}{\rho}\|$ and squaring gives the desired estimate.

When $n = 2$, one computes similarly with $\frac{f}{\rho}$ replaced by $\frac{f}{\rho \ln(\frac{1}{\rho})}$. One needs the extra condition $f(0) = 0$ to extend the estimate to $n = 1$. ☐

Essentially the same proof yields the following variant of the above proposition.

Proposition 3.4. *Let $f \in H^1(\mathbb{R}^n)$, $n > 2$. Let r denote the distance to an affine subspace of codimension $k > 2$. Then*

$$\|df\|^2 \geq \frac{(k-2)^2}{4} \, \|\frac{f}{r}\|^2. \tag{3.5}$$

When $k = 2$ we have

$$\|df\|^2 \geq \frac{1}{4} \|\frac{f}{r \ln(\frac{1}{r})}\|^2. \tag{3.6}$$

3.1.2. Poincaré inequalities for the asymptotically flat ends

Let (M, g) be an asymptotically flat Riemannian manifold of order $\tau > 0$ as defined in Definition 1.1. We modify Proposition 3.1 to a form suitable to the asymptotically flat ends (M_l, Y_l) of M. Let ρ denote the pullback by Y_l of the radial coordinate in $\mathbb{R}^n \setminus B_T(0)$.

Proposition 3.7. *There is a constant $C_l > 0$ so that for all $f \in \mathcal{C}_0^\infty(M_l)$,*

$$\|\frac{f}{\rho}\|^2 \leq \frac{4}{(n-2)^2} \|df\|^2 + C_l \, \|\frac{f}{\rho^{1+\tau/2}}\|^2. \tag{3.8}$$

Proof. In spherical coordinates we have

$$\|f\|^2 = \int_T^\infty \int_{S^{n-1}} |f|^2 \, \rho^{n-1} \mu(\rho, \sigma) \, d\rho d\sigma,$$

where μ satisfies $\mu = 1 + \mathcal{O}(\rho^{-\tau})$ and $|d\mu| = \mathcal{O}(\rho^{-\tau-1})$. Then

$$\begin{aligned}
\|\frac{f}{\rho}\|^2 &= \frac{1}{n-2}\int_T^\infty \int_{S^{n-1}} |f|^2 \ \mu(\rho,\sigma)\, d\rho^{n-2}\, d\sigma \\
&= \frac{-2}{n-2}\int_T^\infty \int_{S^{n-1}} f\frac{\partial f}{\partial \rho}\ \mu(\rho,\sigma)\rho^{n-2} d\rho d\sigma \\
&\quad + \frac{-1}{n-2}\int_T^\infty \int_{S^{n-1}} |f|^2 \frac{\partial \mu(r,\sigma)}{\partial \rho}\ \rho^{n-2} d\rho d\sigma \\
&\le \frac{2}{n-2}\|\frac{f}{\rho|d\rho|}\|\,\|df\| + C\|\frac{f}{\rho^{1+\tau/2}}\|^2,
\end{aligned}$$

for some constant $C > 0$. Asymptotic flatness implies $|d\rho| = 1 + \mathcal{O}(\rho^{-\tau})$. Hence

$$\|\frac{f}{\rho}\|^2 \le \frac{4}{(n-2)^2}\|df\|^2 + C_l\,\|\frac{f}{\rho^{1+\tau/2}}\|^2.$$

for some $C_l > 0$ independent of f. $\square$

3.1.3. Radial estimates near V

We also need Poincaré inequalities near the stratified set V. Together with Kato's inequality, they give growth estimates for spinors in neighborhoods of each of the strata V^{k_b} of V. We derive all these inequalities on tubular neighborhoods in $\mathrm{TRC}(V^{k_b})$ (defined in (2.10)) on which we have a well-behaved radial distance function r_b. Because these estimates use only the radial derivative of a function to control its growth, we will refer to these as *radial* estimates.

Proposition 3.9. *Let $B_\epsilon(Y) \in \mathrm{TRC}(V^{k_b})$. There is a constant $C_Y > 0$ so that for all $f \in \mathcal{C}_0^\infty(B_\epsilon(Y))$,*

$$\|\frac{f}{r_b}\|^2 \le \frac{4}{(k_b-2)^2}\|df\|^2 + C_Y\|\frac{f}{r_b^{1/2}}\|^2 \quad \text{if } k_b > 2, \tag{3.10}$$

and

$$\|\frac{f}{r_b\ln(\frac{1}{r_b})}\|^2 \le 4\|df\|^2 + C_Y\|\frac{f}{r_b^{1/2}\ln^{1/2}(\frac{1}{r_b})}\|^2 \quad \text{if } k_b = 2. \tag{3.11}$$

Proof. The proof is similar to the proof of (3.8). Write

$$\begin{aligned}
\|\frac{f}{r_b}\|^2 &= \int_0^\epsilon \int_{S^{k_b-1}\times Y} |f|^2\, r_b^{n-3}\, m(r_b,\sigma,y)\, dr_b\, d\sigma\, dy \\
&= \frac{1}{k_b-2}\int_0^\epsilon \int_{S^{k_b-1}\times Y} |f|^2\, m(r_b,\sigma,y)\, d(r_b^{k_b-2})\, d\sigma\, dy,
\end{aligned}$$

where m is smooth and multiplicatively bounded. Integrating by parts gives

$$\begin{aligned}
\|\frac{f}{r_b}\|^2 &= -\frac{2}{k_b-2}\int_0^\epsilon \int_{S^{k_b-1}\times Y} f\,\frac{\partial f}{\partial r_b}\, m(r_b,\sigma,y) r_b^{k_b-2}\, dr_b\, d\sigma\, dy \\
&\quad - \frac{1}{k_b-2}\int_0^\epsilon \int_{S^{k_b-1}\times Y} |f|^2\,\frac{\partial m}{\partial r_b}\, r_b^{k_b-2}\, dr_b\, d\sigma\, dy \\
&\le \frac{2}{k_b-2}\|\frac{f}{r_b}\|\,\|df\| + C_Y\|\frac{f}{r_b^{1/2}}\|^2,
\end{aligned}$$

with C_Y a constant depending on the supremum of $\frac{1}{m}$ and $\frac{\partial m}{\partial r_b}$ on $B_\epsilon(Y)$ for all $\epsilon < \epsilon(Y)$. From here, the result follows.

The proof when $k_b = 2$ is similar. $\square$

Remark 3.12. The preceding propositions are written as lower bounds for $\|df\|^2$. In their proofs, however, all derivatives but the (generalized) radial derivatives are discarded. Hence, these estimates can be rewritten as lower bounds for $\|\frac{\partial f}{\partial r_b}\|^2$.

Kato's inequality for a smooth spinor ψ,

$$|d|\psi|| \leq |\nabla\psi| \tag{3.13}$$

on the support of ψ, and the above Poincaré estimate give the following radial estimates for spinors on $M \setminus V$.

Corollary 3.14. *For every $B_\epsilon(Y) \in \mathrm{TRC}(V^{k_b})$ there exists $C_Y > 0$ so that for any spinor ψ in the closure of $\mathcal{C}_0^\infty(B_\epsilon(Y) \setminus V, S)$ with respect the H^1_ρ-norm,*

$$\|\nabla_{\frac{\partial}{\partial r_b}}\psi\|^2 \geq \frac{1}{4}\|\frac{\psi}{r_b}\|^2 - C_Y\|\frac{\psi}{r_b^{1/2}}\|^2 \quad \text{if } k_b > 2, \tag{3.15}$$

and

$$\|\nabla_{\frac{\partial}{\partial r_b}}\psi\|^2 \geq \frac{1}{4}\|\frac{\psi}{r_b\ln(\frac{1}{r_b})}\|^2 - C_Y\|\frac{\psi}{r_b^{1/2}\ln^{1/2}(\frac{1}{r_b})}\|^2 \quad \text{if } k_b = 2. \tag{3.16}$$

3.2. Angular estimates near V^2

Near the codimension two stratum of V, the absence of a zero mode in the Fourier decomposition (2.17) gives us a sharper estimate than the radial estimate (3.16).

We continue with the notation from Section 2.5. Let $W = B_\epsilon(Y) \in \mathrm{TRC}(V^2)$ contractible. Let (r, θ) be the polar coordinates on each normal disk $D_y \subset W$ for $y \in Y$. Let e_θ denote a unit vector in the $\frac{\partial}{\partial\theta}$ direction, and ∇_{e_θ} denote the corresponding covariant derivative on the spinor bundle induced from the Levi-Civita connection. Let $\{e_a\}_{a=1}^n$ be an orthonormal frame of the tangent bundle to M on W. Then ∇_{e_θ} can be expresssed as

$$\nabla_{e_\theta} = e_\theta - \frac{1}{4}\omega_{ab}(e_\theta)c(e_a)c(e_b),$$

where $\omega_{ab}(e_\theta) = g(\nabla_{e_\theta}e_a, e_b)$. The orthonormal frame can be chosen so that the $\omega_{ab}(e_\theta)$ are bounded on W for all $1 \leq a, b \leq n$. Hence we find that on a normal disk in a suitable frame, we have

$$\nabla_{e_\theta} = e_\theta + \mathcal{O}(1).$$

Since the absolute value of the Fourier coefficient of a spinor ψ on $M \setminus V$ restricted to D_y has $\frac{1}{2}$ as a lower bound, we obtain the following *angular estimate*:

Proposition 3.17. *For ψ a spinor on $M \setminus V$ and any $W = B_\epsilon(Y) \in \mathrm{TRC}(V^2)$, there exist a constant $C_Y > 0$ independent of ψ so that on each normal circle $S^1_y(r)$ centered at $y \in Y$ and included in W, we have*

$$\int_{S^1_y(r)} |\nabla_{e_\theta}\psi|^2 d\theta \geq \frac{1}{4}\int_{S^1_y(r)} |\frac{\psi}{r}|^2 d\theta - C_Y\int_{S^1_y(r)} |\frac{\psi}{\sqrt{r}}|^2 d\theta. \tag{3.18}$$

As a consequence,

$$\|\nabla_{e_\theta}\psi\|^2_{L^2(W)} \geq \frac{1}{4}\|\frac{\psi}{r}\|^2_{L^2(W)} - C_Y\|\frac{\psi}{r^{1/2}}\|^2_{L^2(W)}. \tag{3.19}$$

4. The Dirac operator on $M \setminus V$

In this section we derive the first properties of the Dirac operator on $M \setminus V$. We introduce weighted Sobolev spaces, and also the maximal and minimal domains of the Dirac operator viewed as an unbounded operator on L^2-spinors. Then, using the Lichnerowicz formula we derive various properties of the Dirac operator on these spaces, which we then use to verify that certain harmonic spinors satisfy the growth conditions of Witten spinors. We end this section with the proof that the boundary term coming from the asymptotically flat ends in the Lichnerowicz formula applied to a Witten spinor is exactly the mass of the manifold.

Let (M, g) be a Riemannian manifold which is asymptotically flat of order $\tau > 0$ as defined in Definition 1.1. On each asymptotically flat end (M_l, Y_l) we have the induced coordinate system $\{x_i\}$ and the corresponding radial coordinate $\rho(x) = |x|$ obtained by pulling-back the cartesian coordinates on $\mathbb{R}^n$ under Y_l. We extend ρ smoothly over the interior of M so that it is bounded from above by T and from below by 1, and so that it is identically 1 in a neighborhood of V.

We assume that M is nonspin, and let $V \subset K$ be a stratified set given by Theorem 2.7. Let S be the spin bundle corresponding to the maximal spin structure on $M \setminus V$, and let ∇ denote the associated spin connection and D the corresponding Dirac operator. At a point $x \in M \setminus V$, D has the form $D = \sum_{i=1}^n c(e_i)\nabla_{e_i}$ with $\{e_1, \ldots, e_n\}$ any orthonormal frame of tangent vectors at x, ∇ the spin connection on S determined by the Levi-Civita connection of the metric g, and $c(e_i)$ denoting Clifford multiplication by the vector e_i.

4.1. Weighted Sobolev spaces of spinors on $M \setminus V$ and Rellich's compactness

First we define the weighted Sobolev spaces for spinors which will be used to construct the Witten spinors in our main theorem. We also prove a Rellich-type compactness result.

Each open set in $M \setminus V$ is of the form $W = U \setminus V$ with U an open set in M. For each W, let $L^2_\rho(W, S)$ be the completion of $\mathcal{C}^\infty_0(W, S)$ in the norm

$$\|\psi\|_{L^2_\rho} := \|\frac{\psi}{\rho}\|_{L^2}, \tag{4.1}$$

and $H^1_\rho(W, S)$ be the completion of $\mathcal{C}^\infty_0(W, S)$ in the norm

$$\|\psi\|_{H^1_\rho} := \|\nabla\psi\|_{L^2} + \|\psi\|_{L^2_\rho}. \tag{4.2}$$

We have the following version of the Rellich compactness theorem.

Lemma 4.3. *Let U be a bounded open set with smooth boundary in M and let $W = U \setminus V$. The inclusion*

$$H^1_\rho(W, S) \hookrightarrow L^2_\rho(W, S)$$

is compact.

Proof. Let $\{\psi_j\}$ be a sequence of spinors on W, bounded in $H^1_\rho(W,S)$. We need to show that it contains a subsequence which is convergent in $L^2_\rho(W,S)$.

First note that by Kato's inequality $|d|\psi_j||\leq |\nabla\psi_j|$, and the sequence of functions $f_j := |\psi_j|$ forms a bounded sequence in $H^1_0(U)$, the completion of $\mathcal{C}^\infty_0(U)$ in the H^1-norm. By Rellich's compactness theorem for bounded open sets with smooth boundary in complete manifolds, $\{f_j\}$ contains a subsequence, also denoted $\{f_j\}$, which is convergent in $L^2(U)$.

Let $W_1 \subset W_2 \subset \ldots \subset W_k \subset \ldots$ be an exhaustion of W by compact sets with smooth boundary. We show that we can find a subseqeunce of ψ_j which converges in $L^2(W_k,S)$ for each k.

Note that the W_k are compact sets in M which do not intersect V. Let $H^1(W_k,S)$ denote the set of spinors $\psi \in L^2(W_k)$ with $\nabla\psi \in L^2$. By Rellich's compactness theorem, the inclusion $H^1(W_k,S) \hookrightarrow L^2(W_k,S)$ is compact. Since the weight function ρ is multiplicatively bounded on U, $\{\psi_j\} \subset H^1(W_k,S)$ is a bounded sequence for all k. Hence we can find a subsequence $\{\psi_{j,1}\}$ of $\{\psi_j\}$ which converges in $L^2(W_1,S)$. Iterating this we have: given a subsequence $\{\psi_{j,k}\}$ which converges in $L^2(W_k,S)$, we can pass to a new subsequence $\{\psi_{j,k+1}\}$ which converges in $L^2(W_{k+1},S)$. Taking a digonal subsequence, we produce a subsequence, also denoted by $\{\psi_j\}$, which is convergent in $L^2(W_k,S)$ for all k.

To conclude the proof, we need to show that this $\{\psi_j\}$ is a Cauchy subsequence in $L^2_\rho(W,S)$. We have

$$\begin{aligned}\|\psi_j - \psi_l\|_{L^2_\rho(W,S)} &= \|\psi_j - \psi_l\|_{L^2_\rho(W_k,S)} + \|\psi_j - \psi_l\|_{L^2_\rho(W\setminus W_k,S)}\\ &\leq \|\psi_j - \psi_l\|_{L^2_\rho(W_k,S)} + C_U(\|f_j\|_{L^2(U\setminus W_k)} + \|f_l\|_{L^2(U\setminus W_k)}),\end{aligned}$$

for C_U a constant depending on the maximum of ρ on U. Choose $\epsilon > 0$. Since the sequence $\{f_j\}$ is convergent in $L^2(U)$, we can find k large enough and $N_1 > 0$, so that for all $j \geq N_1$, we have $C_U\|f_j\|_{L^2(U\setminus W_k)} \leq \epsilon/3$. Then, since $\{\psi_j\}$ is convergent in $L^2(W_k,S)$, we can find $N > N_1$ so that for all $j,k \geq N$ we have $\|\psi_j - \psi_l\|_{L^2_\rho(W_k,S)} \leq \epsilon/3$. □

4.2. The Dirac operator and the Lichnerowicz formula

The Lichnerowicz formula relates the Dirac Laplacian on $M \setminus V$ to the connection Laplacian on spinors:

$$D^*D = \nabla^*\nabla + \frac{R}{4}. \tag{4.4}$$

Here D^* and ∇^* denote respectively the formal adjoints of the Dirac operator and the spin connection. Since D is a self-adjoint operator, $D^* = D$.

This equality of differential operators gives the *pointwise Lichnerowicz formula*: For all spinors $\psi \in \mathcal{C}^\infty(M \setminus V, S)$ we have

$$|\nabla\psi|^2 + \frac{1}{4}R|\psi|^2 - |D\psi|^2 = \mathrm{div}(W), \tag{4.5}$$

where W is the vector field on $M \setminus V$ defined by

$$\langle W, e\rangle = \langle \nabla_e\psi + c(e)D\psi, \psi\rangle$$

for all $e \in T(M \setminus V)$.

When integrating formula (4.5) on $M \setminus V$, the divergence is expected to introduce boundary terms from the asymptotically flat ends of M and from V. However, when the spinor ψ is in $H^1_\rho(M \setminus V, S)$, this contribution vanishes.

Proposition 4.6. *Let (M, g) be a nonspin Riemannian manifold which is asymptotically flat of order $\tau > 0$. Then, the Dirac operator*

$$D : H^1_\rho(M \setminus V, S) \to L^2(M \setminus V, S),$$

is a bounded linear map which satisfies the integral Lichenerowicz formula

$$\|D\psi\|^2 = \|\nabla\psi\|^2 + \frac{1}{4}(R\psi, \psi). \tag{4.7}$$

Morever, if the scalar curvature is nonnegative, the Dirac operator is injective on $H^1_\rho(M \setminus V, S)$.

Proof. The boundedness of the Dirac operator follows immediately from its definition. To prove the integral Lichnerowicz formula, note that since the metric g is asymptotically flat of order $\tau > 0$, then in the induced frame $\{x_i\}$ on each of the asymptotically flat ends M_l of M we have the scalar curvature

$$\begin{aligned} R &= g^{jk}\left(\partial_i \Gamma^i_{jk} - \partial_k \Gamma^i_{ij} + \Gamma^i_{il}\Gamma^l_{jk} - \Gamma^i_{kl}\Gamma^l_{ij}\right) \\ &= \partial_j(\partial_i g_{ij} - \partial_j g_{ii}) + \mathcal{O}(\rho^{-2\tau-2}), \end{aligned} \tag{4.8}$$

and thus $R = \mathcal{O}(\rho^{-\tau-2})$. From here it follows that each side of the formula (4.7) define continuous functionals on $H^1_\rho(M \setminus V, S)$ which agree on the dense subspace $\mathcal{C}^\infty_0(M \setminus V, S)$.

The injectivity statement is a consequence of the Lichnerowicz formula. Let $\psi \in H^1_\rho(M \setminus V, S)$ so that $D\psi = 0$. Since $R \geq 0$, formula (4.7) implies that ψ is covariantly constant. Since $\psi \to 0$ on the flat ends, it must be identically 0 in order to be in $L^2_\rho(M \setminus V, S)$. $\square$

The weighted Sobolev spaces $H^1_\rho(M \setminus V, S)$ and $L^2_\rho(M \setminus V, S)$ are well adapted for the coercivity results which we prove in Section 8. Because $M \setminus V$ is incomplete, we also need to take care in defining the domain of the Dirac operator. As an operator on the smooth compactly supported sections of $M \setminus V$, the Dirac operator has two natural extensions as an unbounded operator on $L^2(M \setminus V, S)$. The *minimal extension* of D has as domain the *minimal domain* $\mathrm{Dom}_{\min}(D)$, the completion of $\mathcal{C}^\infty_0(M \setminus V, S)$ in the graph norm, $\|\psi\| + \|D\psi\|$. The *maximal extension* has as domain the *maximal domain* $\mathrm{Dom}_{\max}(D)$, which consists of those $\psi \in L^2(M \setminus V, S)$ so that $D\psi \in L^2(M \setminus V, S)$.

We have the following properties of the minimal extension of the Dirac operator on $M \setminus V$.

Corollary 4.9. *Let (M, g) be a nonspin asymptotically flat Riemannian manifold of order $\tau > 0$. Then the minimal extension of the Dirac operator D on $M \setminus V$ satisfies:*

1. $\mathrm{Dom}_{\min}(D) \subset H^1_\rho(M \setminus V, S)$,
2. *Given $\psi \in H^1_\rho(M \setminus V, S)$, $\eta\psi \in \mathrm{Dom}_{\min}(D)$ for all $\eta \in \mathcal{C}^\infty_0(M)$,*
3. *If the scalar curvature is nonnegative, then the null-space of the Dirac operator on the minimal domain is trivial.*

Remark 4.10. In the case of a complete Riemannian spin manifold, the maximal and the minimal extension coincide, [GL, Theorem 1.17].

4.3. Growth conditions for the Witten spinor

We now show that an asymptotically constant harmonic spinor constructed as in Section 1.2 (*assuming existence of a solution to (1.4)*) satisfies the growth conditions of a Witten spinor. This result will be used in the Proof of our Theorem A. We start with the following regularity result on the asymptotically flat ends.

Proposition 4.11. *Let (M, g) be a nonspin Riemannian manifold which is asymptotically flat of order $\tau > 0$. If $\frac{v}{\rho} \in L^2(M \setminus V, S)$ and $Dv \in L^2(M \setminus V, S)$, then $\nabla v \in L^2(M_l, S|_{M_l})$, for every end (M_l, Y_l) of M.*

Proof. Fix an asymptotically flat end M_l, and let η be a smooth cutoff function supported in M_l in the region $\rho \geq L > T$ and identically 1 in the region $\rho \geq 2L$. Next choose a sequence $\{\gamma_j\}$ of smooth cutoff functions on M, compactly supported in the region $\rho \leq 2j$, identically equal to 1 in the region $\rho \leq j$ and so that $|d\gamma_j| \leq \frac{2}{\rho}$. Let $\eta_j := \eta\gamma_j$. Since $\eta_j v \in H^1_\rho(M \setminus V, S)$, the Lichnerowicz formula (4.7) gives

$$\begin{aligned}
\|D(\eta_j v)\|^2 &= \|\nabla(\eta_j v)\|^2 + \frac{1}{4}(R\eta_j v, \eta_j v) \\
&= \|\eta_j \nabla v\|^2 + \||d\eta_j|v\|^2 + 2(\eta_j\nabla v, d\eta_j \otimes v) + \frac{1}{4}(R\eta_j v, \eta_j v) \\
&\geq \frac{1}{2}\|\eta_j\nabla v\|^2 - \||d\eta_j|v\|^2 + \frac{1}{4}(R\eta_j v, \eta_j v).
\end{aligned}$$

We expand the left-hand-side of the above

$$\|D(\eta_j v)\|^2 = \|\eta_j Dv\|^2 + \||d\eta_j|v\|^2 + 2(\eta_j Dv, c(d\eta_j)v),$$

and after rearranging, we obtain

$$\begin{aligned}
\frac{1}{2}\|\eta_j\nabla v\| &\leq \|\eta_j Dv\|^2 + 2(\eta_j Dv, c(d\eta_j)v) + 2\||d\eta_j|v\|^2 - \frac{1}{4}(R\eta_j v, \eta_j v) \\
&\leq \|Dv\|^2 + C\|\frac{v}{\rho}\|^2,
\end{aligned}$$

for $C > 0$ a positive constant independent of j and v. Here we used the fact that since the manifold is asymptotically flat of order $\tau > 0$, the scalar curvature R behaves like $\mathcal{O}(\rho^{-\tau-2})$ on the asymptotically flat ends (see identity (4.8)). Hence we may take the limit as $j \to \infty$ to deduce that $\eta\nabla v \in L^2(M_l, S|_{M_l})$. ◻

Corollary 4.12. *Let (M, g) a nonspin Riemannian manifold which is asymptotically flat of order $\tau > 0$. Let $u \in H^1_\rho(M \setminus V, S)$ be a smooth spinor so that $D^2u \in L^2(M \setminus V, S)$. Then the spinor $v := Du \in \mathrm{Dom}_{\max}(D)$ and $\nabla v \in L^2(M_l, S|_{M_l})$ for each end (M_l, Y_l) of M.*

We call a spinor *constant near infinity* if it is constant on each end M_l with respect to a frame induced by the chosen asymptotically flat coordinate chart (M_l, Y_l). Let ψ_0 be a smooth spinor, constant near infinity and vanishing in a neighborhood of V. Since the coefficients of the spin connection associated to the asymptotically flat metric g differ from

the coefficients of the spin connection associated to the Euclidean metric by terms which decay like $\mathcal{O}(\rho^{-\tau-1})$, it follows that

$$\rho^{\tau+1}|D\psi_0| \quad \text{and} \quad \rho^{\tau+2}|D^2\psi_0| \tag{4.13}$$

are bounded on $M \setminus V$.

As in the case of asymptotically flat *spin* manifolds (see Section 1.2), we construct the Witten spinor in Theorem A by solving $D^2 u = -D\psi_0$, and setting $\psi = \psi_0 + Du$. As a corollary to Corollary 4.12, we see that ψ satisfies the conditions in Definition 1.7 of a Witten spinor:

Corollary 4.14. *Let (M, g) a nonspin Riemannian manifold which is asymptotically flat of order $\tau > \frac{n-2}{2}$. Let ψ_0 be a smooth spinor on $M \setminus V$ which is constant at infinity and supported away from V. Assume that there exists $u \in H^1_\rho(M \setminus V, S)$ so that*

$$D^2 u = -D\psi_0.$$

Let $v = Du$. Then the spinor

$$\psi := v + \psi_0$$

is a Witten spinor.

Proof. By construction ψ is in the null-space of D. From Definition 1.7 of Witten spinors, we only need to check that $v \in H^1_\rho(M \setminus V, S)$. Since the spinor ψ_0 is constant at infinity, it follows that $\rho^{\tau+1}|D\psi_0|$ is bounded. Since $\tau > \frac{n-2}{2}$, $\rho^{\tau+1}|D\psi_0| \in L^2(M \setminus V, S)$, and thus the spinor u satisfies the hypothesis of Proposition 4.12. Hence $v \in H^1_\rho(M \setminus V, S)$. □

4.4. Obtaining the mass from a Witten spinor

For spinors that are not in $H^1_\rho(M \setminus V, S)$, the integral Lichnerowicz formula (4.7) need not hold, because the integration by parts introduces boundary terms. For the Witten spinors, the boundary terms arising from the asymptotically flat ends of M give exactly the mass.

Proposition 4.15. *Let (M, g) be a nonspin Riemannian manifold which is asymptotically flat of order $\tau > \frac{n-2}{2}$. Let ψ_0 be a constant spinor on the asymptotically flat ends of M, with $|\psi_0| \to 1$ at infinity. Let ψ be a Witten spinor on $M \setminus V$ asymptotic to ψ_0. Assume that $\eta\psi \in \mathrm{Dom}_{\min}(D)$ for all $\eta \in \mathcal{C}^\infty_0(M)$. Then*

$$\int_{M\setminus V} |\nabla\psi|^2 + \frac{R}{4}|\psi|^2 = \frac{c(n)}{4}\,\mathrm{mass}(M, g). \tag{4.16}$$

Moreover, if $R \in L^1(M)$ and nonegative, the mass is finite and nonnegative.

Proof. The condition $\eta\psi \in \mathrm{Dom}_{\min}$, $\forall \eta \in C^\infty_0(M)$ implies that, as an ideal boundary, V makes no contribution when integrating the divergence term on the right hand side of the pointwise Lichnerowicz formula (4.5). Since ψ is a Witten spinor, $D\psi = 0$, and thus integrating the pointwise Lichnerowicz formula (4.5) on $M \setminus V$ for this ψ, and applying the divergence theorem to its right-hand-side, we obtain

$$\int_{M\setminus V} |\nabla\psi|^2 + \frac{R}{4}|\psi|^2 = \sum_{l=1}^{L} \lim_{\rho\to\infty} \int_{S^{n-1}_{\rho,l}} \langle \nabla_\nu\psi + c(\nu)D\psi, \psi\rangle d\sigma. \tag{4.17}$$

Here $S^{n-1}_{\rho,l}$ is the sphere of radius ρ in the asymptotically flat coordinate chart (M_l, Y_l) with outward normal vector ν and volume form $d\sigma$. Let $v = \psi - \psi_0$. We split each integral on the right-hand-side of (4.17) into three integrals

$$\int_{S^{n-1}_{\rho,l}} \langle \nabla_\nu \psi_0 + c(\nu) D\psi_0, \psi_0\rangle d\sigma + \int_{S^{n-1}_{\rho,l}} \langle \nabla_\nu v + c(\nu)\nabla_\nu v, \psi_0\rangle d\sigma + \int_{S^{n-1}_{\rho,l}} \langle \nabla_\nu \psi, v\rangle d\sigma. \quad (4.18)$$

As follows from Witten's proof (see [Bar, PT, LP]), the first integral converges to $\frac{c(n)}{4}\,\mathrm{mass}(M_l, g)$ as $\rho \to \infty$. It remains to prove that the other two converge to 0 as $\rho \to \infty$.

In the second integral, we rewrite the integrand in an orthonormal frame $\{e_i\}_i$, with $e_1 = \nu$ as

$$\begin{aligned}\sum_{j=2}^{n} \langle \frac{1}{2}[c(\nu), c(e_j)]\nabla_{e_j} v, \psi_0\rangle =& \sum_{j=2}^{n} e_j \langle \frac{1}{2}[c(\nu), c(e_j)]v, \psi_0\rangle - \sum_{j=2}^{n} \langle \frac{1}{2}[c(\nu)c(\nabla_{e_j} e_j)]v, \psi_0\rangle \\ &- \sum_{j=2}^{n} \langle \frac{1}{2}[c(\nabla_{e_j}\nu), c(e_j)]v, \psi_0\rangle - \sum_{j=2}^{n} \langle \frac{1}{2}[c(\nu), c(e_j)]v, \nabla_{e_j}\psi_0\rangle.\end{aligned} \quad (4.19)$$

We recognize the first two sums on the right hand side of Equation 4.19 as the divergence (on the sphere) of the vector field

$$U := \sum_{j=2}^{n} \langle \frac{1}{2}[c(\nu), c(e_j)]v, \psi_0\rangle e_j.$$

Hence these terms integrate to zero on the sphere. The third term on the right hand side of Equation 4.19 vanishes due to the symmetry of the second fundamental form A of $S^{n-1}_{\rho,l} \subset M_l$. Explicitly, we have

$$\sum_{j=2}^{n} [c(\nabla_{e_j}\nu), c(e_j)] = \sum_{i,j=2}^{n} A(e_j, e_i)[c(e_i), c(e_j)] = 0.$$

Thus, it remains to evaluate

$$\begin{aligned}\lim_{\rho\to\infty} \int_{S^{n-1}_{\rho,l}} \sum_{j=2}^{n} \langle \frac{1}{2}[c(\nu), c(e_j)]v, \nabla_{e_j}\psi_0\rangle d\sigma + \lim_{\rho\to\infty} \int_{S^{n-1}_{\rho,l}} \langle \nabla_\nu \psi, v\rangle d\sigma \\ = \lim_{\rho\to\infty} \int_{S^{n-1}_{\rho,l}} \left(\langle v, \nabla_\nu \psi_0 + c(\nu) D\psi_0\rangle + \langle \nabla_\nu \psi, v\rangle \right) d\sigma.\end{aligned} \quad (4.20)$$

Since ψ_0 is asymptotically constant, $\nabla\psi_0 = \mathcal{O}(\rho^{-\tau-1})$, and since $\tau > \frac{n-2}{2}$, $\nabla\psi_0 \in L^2(M \setminus V, S)$. Moreover since ψ is a Witten spinor, $\frac{v}{\rho} \in L^2(M \setminus V, S)$ and $\nabla\psi \in L^2(M_l)$ for all asymptotically flat ends (M_l, Y_l). Hence, as a function of ρ

$$\int_{S^{n-1}_{\rho}} \left(\langle v, \nabla_\nu \psi_0 + c(\nu) D\psi_0\rangle + \langle \nabla_\nu \psi, v\rangle \right) d\sigma \in L^1([T, \infty), \frac{d\rho}{\rho}),$$

which implies that the integral converges to 0 as $\rho \to \infty$. $\square$

5. Growth estimates near V for spinors in the minimal domain

We now turn to the study of spinors near V. This will occupy the next two sections. In this section, we derive growth estimates near each stratum of V for spinors in the minimal domain. For this we use the Lichnerowicz formula together with the radial and angular estimates near V obtained in Section 3. We also give conditions guaranteeing a spinor is in the minimal domain.

Lemma 5.1. *Let $u \in \mathrm{Dom}_{\min}(D)$ with u supported in $W \in \mathrm{TRC}(V^2)$. Then*

$$\|Du\|^2 \geq \frac{1}{4}\|\frac{u}{r}\|^2 + \frac{1}{4}\|\frac{u}{r\ln(\frac{1}{r})}\|^2 - C_W\|\frac{u}{r^{1/2}}\|, \tag{5.2}$$

for some constant $C_W > 0$, depending on W and independent of u.

Proof. Since $u \in \mathrm{Dom}_{\min}(D)$, the Lichnerowicz formula (4.7) gives

$$\|Du\|^2 \geq \|\nabla u\|^2.$$

The estimate (3.19) for the angular derivative and the estimate (3.16) for the radial derivative combine to give the desired estimate. □

Lemma 5.3. *Let $u \in \mathrm{Dom}_{\min}(D)$ be supported in $W \in \mathrm{TRC}(V^{k_b})$ with $k_b > 2$. Then*

$$\|Du\|^2 \geq \frac{(k_b-2)^2}{4}\|\frac{u}{r_b}\|^2 - C_W\|\frac{u}{r_b^{1/2}}\|^2, \tag{5.4}$$

where $C_W > 0$ is a positive constant depending on W and independent of u.

Proof. Since u is supported in $W \in \mathrm{TRC}(V^{k_b})$ with $k_b > 2$, the Lichnerowicz formula and the radial estimate (3.15) give the desired estimate. □

As a consequence of the previous two Lemmas, we have

Corollary 5.5. *Let $u \in \mathrm{Dom}_{\min}(D)$. Then for any b so that V^{k_b} is nonempty,*

$$\frac{u}{r_b} \in L^2(W \setminus V, S|_{W\setminus V}) \quad \textit{for all} \quad W \in \mathrm{TRC}(V^{k_b}). \tag{5.6}$$

We also have a useful partial converse result.

Lemma 5.7. *Let $u \in \mathrm{Dom}_{\max}(D)$. If for all b so that V^{k_b} is nonempty and for all $W \in \mathrm{TRC}(V^{k_b})$, $\frac{u}{r_b} \in L^2(W)$, then $u \in \mathrm{Dom}_{\min}(D)$.*

Proof. It suffices to construct a sequence of spinors in $\mathrm{Dom}_{\min}(D)$ that converge to u in the graph norm of D. Note that since the closure of the asymptotically flat ends of the manifold are complete, $u \in \mathrm{Dom}_{\max}(D)$ implies that $\chi u \in \mathrm{Dom}_{\min}(D)$ for any smooth, C^1-bounded χ supported away from V. Therefore, we can construct our desired sequence, by using cutoff functions supported in the complement of smaller and smaller neighborhoods of V.

Let $\{\gamma_m\}_m$ be a sequence of smooth functions $\gamma_m : (0,1) \to \mathbb{R}$ satisfying

1. $0 \leq \gamma_m(t) \leq 1$ vanishes in a neighborhood of 0,
2. $\gamma_m(t) = 1$ for $t > 2e^{-e^m}$, and
3. $|d\gamma_m| < \frac{1}{t\ln(\frac{1}{t})}$, $\forall m$.

Piecewise differentiable γ_m satisfying these properties are easily constructed by letting $\gamma_m'(t)$ be the product of $\frac{1}{t\ln(\frac{1}{t})}$ and the characteristic function of the interval $[e^{-e^{m+1}}, e^{-e^m}]$. Smoothing the characteristic function of the interval suffices to construct smooth γ_m.

Let $2 = k_2 < k_3 < \cdots < k_p$ denote the codimensions for which V^{k_j} is nonempty. The highest codimension stratum, V^{k_p}, is a compact subset of M. Take $W^{k_p} := B_{\epsilon_p}(V^{k_p}) \in \mathrm{TRC}(V^2)$ for some $\epsilon_p > 0$. Extend r_p from W^{k_p} to M as a smooth, nonnegative eventually constant function, bounded above by $\frac{1}{2}$. Then $\{\gamma_m(r_p)u\}_m$ converges in L^2 to u. By hypothesis, $\frac{u}{r_p} \in L^2(W^{k_p})$, and therefore $|[D, \gamma_m(r_p)]u| \leq \frac{|u|}{r_p \ln(\frac{1}{r_p})} \in L^2$. Since

$$D(\gamma_m(r_p)u) = [D, \gamma_m(r_p)]u + \gamma_m(r_p)Du,$$

and $\gamma_m \to 1$ pointwise, Lebesgue's dominated convergence theorem implies $[D, \gamma_m(r_p)]u$ converges to 0 in L^2-norm, and $D(\gamma_m(r_p)u)$ converges in L^2 to Du as $m \to \infty$.

Consider now $\gamma_m(r_p)u$ for m fixed. By construction, this vanishes on $B_{\epsilon(m,p)}(V^{k_p}) \subset W^{k_p}$ for some $\epsilon(m,p) > 0$. Observe that for any $\epsilon > 0$ such that $B_\epsilon(V^{k_{p-1}} \setminus B_{\frac{1}{2}\epsilon(m,p)}(V^{k_p})) \in \mathrm{TRC}(V^{k_{p-1}})$, $B_\epsilon(V^{k_{p-1}} \setminus B_{\frac{1}{2}\epsilon(m,p)}(V^{k_p})) \cup B_{\epsilon(m,p)}(V^{k_p})$ is a neighborhood of $\overline{V^{k_{p-1}}}$. Extend r_{p-1} from $B_\epsilon(V^{k_{p-1}} \setminus B_{\frac{1}{2}\epsilon(m,p)}(V^{k_p}))$ as a smooth, nonnegative, eventually constant function on M, bounded above by $\frac{1}{2}$. Then each of the spinors $\{\gamma_\mu(r_{p-1})\gamma_m(r_p)u\}_\mu$ is supported outside a neighborhood of $\overline{V^{k_{p-1}}}$. For ϵ fixed and μ sufficiently large $[D, \gamma_\mu(r_{p-1})]\gamma_m(r_p)u$ is supported in $B_\epsilon(V^{k_{p-1}} \setminus B_{\frac{1}{2}\epsilon(m,b)}(V^{k_p}))$ and satisfies

$$|[D, \gamma_\mu(r_{p-1})]\gamma_m(r_p)u| \leq \frac{|\gamma_m(r_p)u|}{r_{p-1}\ln(\frac{1}{r_{p-1}})} \in L^2(B_\epsilon(V^{k_{p-1}} \setminus B_{\frac{1}{2}\epsilon(m,b)}(V^{k_p}))).$$

Hence we may again apply the Lebesgue dominated convergence theorem to conclude that $\gamma_\mu(r_{p-1})\gamma_m(r_p)u \overset{L^2}{\to} \gamma_m(r_p)u$ and $D(\gamma_\mu(r_{p-1})\gamma_m(r_p)u) \overset{L^2}{\to} D(\gamma_m(r_p)u)$ as $\mu \to \infty$.

Proceeding inductively backwards on k_b, we construct a sequence of spinors in the minimal domain of D which converge to u in the graph norm of D. Hence $u \in \mathrm{Dom}_{\min}(D)$. $\square$

6. Improved integral estimates for harmonic spinors near V

For harmonic spinors, we can improve the growth estimates of Section 5. In this section, we use the lower bound estimates derived in Section 5 to obtain weighted integral estimates for spinors u in the minimal domain of D satisfying $D^2u = 0$ in a neighborhood of V, and for the spinors $v = Du$. We conclude with a result which gives a sufficient condition for v to be in the minimal domain of D.

Our main tool is a technique of Agmon, [Ag], which we now present. If $u \in \mathrm{Dom}_{\min}(D)$, then $fu \in \mathrm{Dom}_{\min}(D)$ for any bounded, piecewise differentiable function f, with bounded derivative. Hence we have the following identity

$$(D^2u, f^2u) = \|D(fu)\|^2 - \|[D, f]u\|^2. \tag{6.1}$$

Combining this identity with the Lichnerowicz formula, we have

$$(D^2u, f^2u) = \|\nabla(fu)\|^2 + \frac{1}{4}(Rfu, fu) - \|[D, f]u\|^2. \tag{6.2}$$

We refer to either of the expressions (6.1) or (6.2) as the *Agmon identity.* In this section we apply this identity, together with the radial and angular estimates which we derived in Section 3, to obtain our desired estimates.

6.1. Estimates near the codimension 2 stratum of V

We prove first estimates near the codimension 2 stratum V^2.

Lemma 6.3. *If* $u \in \mathrm{Dom}_{\min}(D)$ *with* $D^2u = 0$ *in a neighborhood of* $\overline{V^2}$*, then*

$$\frac{u}{r^{3/2}\ln^{1/2+a}(\frac{1}{r})} \in L^2(W \setminus V, S|_{W\setminus V}) \tag{6.4}$$

for all $a > 0$ *and for all* $W \in \mathrm{TRC}(V^2)$*.*

Proof. From the definition (2.10), given $W \in \mathrm{TRC}(V^2)$, there exists a relatively compact subset Y of V^2 and $\epsilon < \epsilon(Y)$ so that $W = B_\epsilon(Y)$. Without loss of generality we can assume that $D^2u = 0$ in an open set $W' \in \mathrm{TRC}(V^2)$ containing $\overline{W}$.

Let $\zeta \in C_0^\infty(W')$ with $0 \leq \zeta \leq 1$ and $\zeta \equiv 1$ on W. Given $a > 0$, we define the sequence of functions

$$\mu_m(r) := \begin{cases} \zeta r^{-1/2}\ln^{-a}(\frac{1}{r}) & \text{for } r > \frac{1}{m} \\ \zeta m^{1/2}\ln^{-a}(m) & \text{for } r < \frac{1}{m}. \end{cases}$$

Then, the Agmon identity (6.1) applied to the functions μ_m,

$$0 = (D^2u, \mu_m^2 u) = \|D(\mu_m u)\|^2 - \|[D, \mu_m]\|^2,$$

together with the estimate (5.2) give

$$0 \geq \frac{1}{4}\|\frac{\mu_m u}{r}\|^2 + \frac{1}{4}\|\frac{\mu_m u}{r\ln(\frac{1}{r})}\|^2 - \|[D, \mu_m]u\|^2 - C_{W'}\|\frac{\mu_m u}{r^{1/2}}\|^2, \tag{6.5}$$

with the constant $C_{W'}$ depending on W'. Expanding $[D, \mu_m]u$, we see that for large enough m,

$$\|[D, \mu_m]u\|^2 \leq \frac{1}{4}\|\frac{\mu_m u}{r}\|^2 - a\|\frac{\mu_m u}{r\ln^{1/2}(\frac{1}{r})}\|^2 + a^2\|\frac{\mu_m u}{r\ln(\frac{1}{r})}\|^2 + \|d\zeta\|^2_{L^\infty(W')}\|\frac{u}{r^{1/2}}\|^2_{L^2(W')} + C_{\zeta,\epsilon}\|\frac{u}{r}\|^2_{L^2(W')},$$

with $C_{\zeta,\epsilon}$ a constant depending on $\|d\zeta\|_{L^\infty(W')}$ and ϵ. By Corollary 5.5, $\|\frac{u}{r}\|^2_{L^2(W')}$ is finite. Plugging this expansion back into (6.5) gives

$$(C_{\zeta,\epsilon} + C_{W'})\|\frac{u}{r}\|^2_{L^2(W')} + \|d\zeta\|^2_{L^\infty(W')}\|\frac{u}{r^{1/2}}\|^2_{L^2(W')} \geq a\|\frac{\mu_m u}{r\ln^{1/2}(\frac{1}{r})}\|^2 + \frac{1}{4}\|\frac{\mu_m u}{r\ln(\frac{1}{r})}\|^2 - a^2\|\frac{\mu_m u}{r\ln(\frac{1}{r})}\|^2.$$

Taking the limit as $m \to \infty$, it follows that

$$\frac{u}{r^{3/2}\ln^{1/2+a}(\frac{1}{r})} \in L^2(W \setminus V, S|_{W\setminus V}),$$

for all $a > 0$. ∎

Lemma 6.6. *Let $v = Du$, with u as in the previous Lemma. Then*

$$\frac{v}{r^{1/2}\ln^{1/2+a}(\frac{1}{r})} \in L^2(W \setminus V, S|_{V\setminus W}) \tag{6.7}$$

for all $a > 0$ and for all $W \in \mathrm{TRC}(V^2)$.

Proof. Let W' and ζ be as in the proof of Lemma 6.3. We modify the sequence of functions in the proof of the preceeding lemma to

$$\mu_m(r) := \begin{cases} \zeta r^{-1/2}\ln^{-1/2-a}(\frac{1}{r}) & \text{for } r < \frac{1}{m} \\ \zeta m^{1/2}\ln^{-1/2-a}(m) & \text{for } r > \frac{1}{m}. \end{cases}$$

The Agmon identity (6.1) applied to μ_m gives

$$\|D(\mu_m u)\|^2 = \|[D, \mu_m]u\|^2.$$

The preceeding lemma implies that $\|[D, \mu_m]u\|^2$ is uniformly bounded as $m \to \infty$. Hence, taking the limit as $m \to \infty$ we obtain

$$D(\frac{\zeta u}{r^{1/2}\ln^{1/2+a}(\frac{1}{r})}) \in L^2(W \setminus V, S|_{W\setminus V}).$$

Writing

$$D(\frac{\zeta u}{r^{1/2}\ln^{1/2+a}(\frac{1}{r})}) = \frac{\zeta Du}{r^{1/2}\ln^{1/2+a}(\frac{1}{r})} + [D, \frac{\zeta(r)}{r^{1/2}\ln^{1/2+a}(\frac{1}{r})}]u,$$

expresses $\frac{\zeta v}{r^{1/2}\ln^{1/2+a}(\frac{1}{r})}$ as the difference of L^2-sections. Hence it is square integrable. □

6.2. Estimates near the higher codimension strata of V

Now we prove similar estimates to (6.4) and (6.7) near the higher codimension strata of V. These estimates will be in terms of r_b, the distance to the stratum V^{k_b}.

Lemma 6.8. *If $u \in \mathrm{Dom}_{\min}(D)$ and $D^2u = 0$ in a neighborhood of $\overline{V^{k_b}}$ in $M \setminus V$ with $k_b > 2$, then*

$$\frac{u}{r_b^{k_b/2}\ln^{1/2+a}(\frac{1}{r_b})} \in L^2(W \setminus V, S|_{W\setminus V}) \tag{6.9}$$

for all $W \in \mathrm{TRC}(V^{k_b})$ and for all $a > 0$.

Proof. The proof is similar to the proof of Lemma 6.3, with (5.4) replacing (5.2). It consists of two steps. In the first step we show that

$$\frac{u}{r_b^{\alpha}} \in L^2(W \setminus V, S|_{W\setminus V}) \tag{6.10}$$

for all $\alpha < k_b/2$ and for all $W \in \mathrm{TRC}(V^{k_b})$. In the second step we prove (6.9).

Step 1:

Let $W \in \mathrm{TRC}(V^{k_b})$. By (2.10), this means that there exists Y a relatively compact subset of V^{k_b} and $\epsilon < \epsilon(Y)$ so that $W = B_\epsilon(Y)$. Without loss of generality, we can assume that $D^2u = 0$ in an open set $W' \in \mathrm{TRC}(V^{k_b})$ containing $\overline{W}$.

Let $\zeta \in C_0^\infty(W')$ with $0 \le \zeta \le 1$ and $\zeta \equiv 1$ on W. Then the Agmon identity (6.1) applied to the sequence of functions $\mu_m \zeta$ defined as

$$\mu_m(r_b) := \begin{cases} r_b^{-\alpha} & \text{for } r_b < \frac{1}{m} \\ m^\alpha & \text{for } r_b \ge \frac{1}{m}, \end{cases}$$

gives

$$0 = \|D(\mu_m \zeta u)\|^2 - \|[D, \mu_m \zeta]u\|^2.$$

Applying (5.4) to the first term, we obtain

$$0 \ge \frac{(k_b-2)^2}{4} \|\frac{\mu_m \zeta u}{r_b}\|^2 - C_{W'} \|\frac{\mu_m \zeta u}{r_b^{1/2}}\| - \|[D, \mu_m \zeta]u\|^2 \tag{6.11}$$

with $C_{W'}$ a constant depending on W'. Expanding $[D, \mu_m \zeta]u$, we have

$$\|[D, \mu_m \zeta]u\|^2 \le \alpha^2 \|\frac{\mu_m \zeta u}{r_b}\|^2 + \|d\zeta\|^2_{L^\infty(W')} \|\mu_m u\|^2_{L^2(W')} + 2\alpha \|d\zeta\|_{L^\infty(W')} \|\frac{\chi_m \chi_{d\zeta} u}{r_b^{\alpha+1/2}}\|^2.$$

Here χ_m denotes the characteristic function of $r > \frac{1}{m}$ and $\chi_{d\zeta}$ the characteristic function of the support of $d\zeta$. Plugging this into (6.11), we obtain

$$\begin{aligned} \|d\zeta\|^2_{L^\infty(W')} \|\mu_m u\|^2_{L^2(W')} + 2\alpha \|d\zeta\|_{L^\infty(W')} \|\frac{\chi_m \chi_{d\zeta} u}{r_b^{\alpha+\frac{1}{2}}}\|^2 + C_{W'} \|\frac{\mu_m u}{r_b^{1/2}}\|^2_{L^2(W')} \\ \ge \frac{(k_b-2)^2}{4} \|\frac{\mu_m \zeta u}{r_b}\|^2 - \alpha^2 \|\frac{\mu_m \zeta u}{r_b}\|^2. \end{aligned}$$

Assuming that $\frac{u}{r_b^{\alpha+1/2}} \in L^2(W' \setminus V, S|_{W' \setminus V})$, it follows that the the left-hand-side above is bounded by a constant independent of m. Hence we can take the limit as $m \to \infty$ and conclude

$$\frac{u}{r_b^{\alpha+1}} \in L^2(W \setminus V, S|_{W \setminus V})$$

as long as $\alpha < (k_b - 2)/2$.

By Corollary 5.5, we know that $\frac{u}{r_b} \in L^2(W', S|_{W' \setminus V})$. Starting from here, and bootstrapping using the above argument, we obtain (6.10).

Step 2:

We have the same argument as in the proof of the first step, only that now we take

$$\mu_m(r_b) := \begin{cases} r_b^{-(k_b-2)/2} \ln^{-a}(\frac{1}{r_b}) & \text{for } r_b < \frac{1}{m} \\ m^{(k_b-2)/2} \ln^{-a}(m) & \text{for } r_b > \frac{1}{m}, \end{cases}$$

for $a > 0$. This time, expanding $[D, \mu_m \zeta]u$, we obtain

$$\begin{aligned} \|[D, \mu_m \zeta]u\|^2 &= \|[D, \mu_m]\zeta u\|^2 + 2([D, \mu_m]\zeta u, \mu_m [D, \zeta]u) + \|\mu_m [D, \zeta]u\|^2 \\ &\le \frac{(k_b-2)^2}{4} \|\frac{\mu_m \zeta u}{r_b}\|^2 - (k_b-2)a \|\frac{\chi_m \mu_m \zeta u}{r_b \ln^{1/2}(\frac{1}{r_b})}\|^2 + a^2 \|\frac{\mu_m \zeta u}{r_b \ln(\frac{1}{r_b})}\|^2 \\ &\quad + \frac{k_b-2}{2} \|d\zeta\|_{L^\infty(W')} \|\frac{\mu_m \chi_m \zeta^{1/2} u}{r_b^{1/2}}\|^2 + a \|d\zeta\|_{L^\infty(W')} \|\frac{\mu_m \chi_m \zeta^{1/2} u}{r_b^{1/2} \ln^{1/2}(\frac{1}{r_b})}\|^2 \\ &\quad + \|d\zeta\|^2_{L^\infty(W')} \|\mu_m \chi_{d\zeta} u\|^2. \end{aligned}$$

Plugging this into (6.11), we obtain

$$\begin{aligned}(k_b-2)a\ \|\frac{\chi_m\mu_m\zeta u}{r_b\ln^{1/2}(\frac{1}{r_b})}\|^2 - a^2\|\frac{\mu_m\zeta u}{r_b\ln(\frac{1}{r_b})}\|^2 \\ \le \frac{k_b-2}{2}\|d\zeta\|_{L^\infty(W')}\|\frac{\mu_m\chi_m\zeta^{1/2}u}{r_b^{1/2}}\|^2 + a\|d\zeta\|_{L^\infty(W')}\|\frac{\mu_m\chi_m\zeta^{1/2}u}{r_b^{1/2}\ln^{1/2}(\frac{1}{r_b})}\|^2 \\ + \|d\zeta\|^2_{L^\infty(W')}\|\mu_m\chi_{d\zeta}u\|^2 + C_{W'}\|\frac{\mu_m\zeta u}{r_b^{1/2}}\|.\end{aligned}$$

By the first step, we know that the right-hand-side is bounded by a constant independent of m. Therefore we can take limit as $m\to\infty$ and conclude

$$\frac{u}{r_b^{k_b/2}\ln^{1/2+a}(\frac{1}{r_b})} \in L^2(W\setminus V, S|_{W\setminus V})$$

for all $a>0$. ☐

For $v = Du$ we have now the following estimate near the higher codimension strata.

Lemma 6.12. *Let $v = Du$, with u as in Lemma 6.8. Then*

$$\frac{v}{r_b^{(k_b-2)/2}\ln^{1/2+a}(\frac{1}{r_b})} \in L^2(W\setminus V, S|_{W\setminus V}) \tag{6.13}$$

for all $W\in \mathrm{TRC}(V^{k_b})$ and all $a>0$.

The proof is similar to the proof of Lemma 6.6, using the result of Lemma 6.8.

6.3. A sufficient condition for v to be in $\mathrm{Dom}_{\min}(D)$

The following result gives a sufficient condition for the spinor $v = Du$, considered above, to be in the minimal domain of D.

Proposition 6.14. *Let $v = Du$ with $u\in \mathrm{Dom}_{\min}(D)$ and $D^2u = 0$ in a neighborhood of V. If $\frac{v}{r^{1/2}\ln^{1/2}(\frac{1}{r})} \in L^2(W\setminus V, S|_{W\setminus V})$ for all $W\in TRC(V^2)$, then $\eta v\in \mathrm{Dom}_{\min}(D)$, for all $\eta\in \mathcal{C}_0^\infty(M)$.*

Note that the hypothesis is stronger than the estimate we obtained in Lemma 6.6; in particular, it rules out $|v|$ growing like $r^{-1/2}$ near V^2.

Proof. We show that v satisfies the hypotheses of Lemma 5.7. By Lemma 6.12 we have the desired estimates near the higher codimension strata V^{k_b} with $k_b>2$. The only difficult stratum is V^2. Let $W\in \mathrm{TRC}(V^2)$, and let $W'\in \mathrm{TRC}(V^2)$ containing $\overline{W}$. Let $\zeta\in C_0^\infty(W')$ with $0\le\zeta\le 1$ and $\zeta\equiv 1$ on W. Let $\{\gamma_m(r)\}_m$ be the sequence of cutoff functions described in Lemma 5.7. Define another sequence of bounded, piecewise differentiable functions by

$$\mu_j(r) := \begin{cases} j^{1/2}r^{1/2} & \text{for } r<\frac{1}{j} \\ 1 & \text{for } r\ge \frac{1}{j}, \end{cases}$$

and let χ_j denote the characteristic function of the support of $d\mu_j$. The estimate (5.2) applied to $\gamma_m\mu_j\zeta v \in \mathrm{Dom}_{\min}(D)$ gives

$$\|D(\gamma_m\mu_j\zeta v)\|^2 \geq \frac{1}{4}\|\frac{\gamma_m\mu_j\zeta v}{r}\|^2 - C_{W'}\|\frac{\gamma_m\mu_j\zeta v}{r^{1/2}}\|^2. \tag{6.15}$$

On the other hand, since $Dv = 0$ on the support of ζ,

$$\begin{aligned}\|D(\gamma_m\mu_j\zeta v)\|^2 &= \|[D,\gamma_m]\mu_j\zeta v + \frac{\chi_j}{2r}c(e_r)\gamma_m\mu_j\zeta v + \gamma_m\mu_j[D,\zeta]v\|^2 \\ &= \|[D,\gamma_m]\mu_j\zeta v\|^2 + \frac{1}{4}\|\frac{\gamma_m\chi_j\mu_j\zeta v}{r}\|^2 + \|\gamma_m\mu_j[D,\zeta]v\|^2 \\ &\quad + 2([D,\gamma_m]\mu_j\zeta v, \gamma_m\mu_j[D,\zeta]v + \frac{\chi_j}{2r}c(e_r)\gamma_m\mu_j\zeta v) + 2(\gamma_m\mu_j[D,\zeta]v, \frac{\chi_j}{2r}c(e_r)\gamma_m\mu_j\zeta v).\end{aligned}$$

Since $\frac{v}{r^{1/2}\ln^{1/2}(\frac{1}{r})} \in L^2(W')$ by hypothesis, the Lebesgue dominated convergence theorem implies that

$$\lim_{m\to\infty}\Big(([D,\gamma_m]\mu_j\zeta v, \gamma_m\mu_j[D,\zeta]v + \frac{\chi_j}{2r}c(e_r)\gamma_m\mu_j\zeta v) + \|[D,\gamma_m]\mu_j\zeta v\|^2\Big) = 0.$$

Hence, substituting the above expression for $\|D(\gamma_m\mu_j\zeta v)\|^2$ into (6.15) and taking the limit as $m \to \infty$ yields

$$\|\mu_j[D,\zeta]v\|^2 + 2(\mu_j[D,\zeta]v, \frac{\chi_j}{2r}c(e_r)\mu_j\zeta v) \geq \frac{1}{4}\|\frac{(1-\chi_j)\mu_j\zeta v}{r}\|^2 - C_{W'}\|\frac{\mu_j\zeta v}{r^{1/2}}\|^2.$$

Taking now the limit as $j \to \infty$ gives

$$\frac{1}{4}\|\frac{\zeta v}{r}\|^2 \leq C_{W'}\|\frac{\zeta v}{r^{1/2}}\|^2 + \|[D,\zeta]v\|^2,$$

and therefore $\frac{v}{r} \in L^2(W, S|_{W\setminus V})$. We may now apply Lemma 5.7 to deduce $v \in \mathrm{Dom}_{\min}(D)$. $\square$

7. Estimates on the asymptotically flat ends of M

In this section, we show that if $u \in H^1_\rho(M\setminus V, S)$ is a smooth spinor with D^2u satisfying certain decay estimates, then u satisfies weighted integral estimates on the asymptotically flat ends of M. These results are used to prove Theorem A. Such estimates are not new (see [Bar, PT]), but we choose to obtain them here using Agmon's identity in a manner similar to the proof in Section 6 of our weighted integral estimates near V.

We first record the extension of Agmon's identity (6.1) from $\mathrm{Dom}_{\min}(D)$ to $H^1_\rho(M\setminus V, S)$.

Lemma 7.1. *Let $u \in H^1_\rho(M\setminus V, S)$ be a smooth spinor. Then for any bounded, piecewise differentiable function f, with bounded derivatives, satisfying $|df|u \in L^2(M\setminus V, S)$ and the pointwise inner-product $\langle D^2u, f^2u\rangle(x) \in L^1(M\setminus V)$, we have the following identities in L^2-norm on $M\setminus V$:*

$$(D^2u, f^2u) = \|D(fu)\|^2 - \|[D,f]u\|^2 = \|\nabla(fu)\|^2 + \frac{1}{4}(Rfu, fu) - \|[D,f]u\|^2. \tag{7.2}$$

Proof. Since $u \in H^1_\rho(M\setminus V, S)$, $\eta u \in \mathrm{Dom}_{\min}(D)$ for all $\eta \in \mathcal{C}^\infty_0(M)$. Choose a sequence of smooth compactly supported functions $\eta_j \in \mathcal{C}^\infty_0(M)$ so that $0 \leq \eta_j \leq 1$ is supported in the

region where $\rho(x) \leq 2j$, is identically 1 where $\rho(x) \leq j$, and satisfies $|d\eta_j| \leq \frac{2}{\rho}$. Then by Agmon's identity (6.1), we have

$$\begin{aligned}(D^2u, f^2\eta_j^2 u) &= \|D(\eta_j f u)\|^2 - \|[D, \eta_j f]u\|^2 \\ &= \|\eta_j D(fu) + [D, \eta_j]fu\|^2 - \|[D, \eta_j f]u\|^2.\end{aligned}$$

Since $\frac{u}{\rho}$ and $D(fu)$ are both in $L^2(M \setminus V, S)$, and since $|d\eta_j| < \frac{2}{\rho}$, the Lebesgue dominated convergence theorem gives

$$2(\eta_j D(fu), [D, \eta_j]fu) + \|[D, \eta_j]fu\|^2 \to 0 \quad \text{as } j \to \infty.$$

Hence, we may thus take the limit as $j \to \infty$ of the preceding equalities to get

$$(D^2u, f^2u) = \|D(fu)\|^2 - \|[D, f]u\|^2.$$

The second part of the identity (7.2) follows similarly, using the Lichnerowicz formula (4.7).

□

We first illustrate the use of this Agmon identity to derive, in our context, the standard result that $\rho^{a+1}D^2u \in L^2$ implies $\rho^{a-1}u \in L^2$.

Proposition 7.3. *Let (M^n, g) be a nonspin Riemannian manifold which is asymptotically flat of order $\tau > 0$ and has nonnegative scalar curvature. Let $u \in H^1_\rho(M \setminus V, S)$ be a smooth spinor. If there exists a positive real number $0 \leq a < \frac{n-2}{2}$ so that $\rho^{a+1}D^2u \in L^2(M_l, S|_{M_l})$ for each asymptotically flat end M_l of M, then*

$$\rho^{a-1}u \in L^2(M \setminus V, S). \tag{7.4}$$

Proof. Let η be a cutoff function, $0 \leq \eta \leq 1$, supported in the region of M_l where $\rho(x) \geq L$ and identically 1 in the region where $\rho(x) \geq 2L$, for some L to be fixed later. For each positive integer m, consider

$$\mu_m(\rho) := \begin{cases} \rho^a & \text{if } \rho \leq m, \\ m^a & \text{if } \rho > m. \end{cases}$$

Then (7.2) applied to the functions $\mu_m\eta$, gives

$$(D^2u, \mu_m^2\eta^2u) = \|D(\mu_m\eta u)\|^2 - \||[D, \mu_m\eta]u\|^2,$$

and since we assume that the scalar curvature is nonnegative,

$$(D^2u, \mu_m^2\eta^2u) \geq \|\nabla(\mu_m\eta u)\|^2 - \|[D, \mu_m\eta]u\|^2.$$

Applying the Poincaré inequality (3.8) on the asymptotically flat end M_l of M, we obtain

$$(D^2u, \mu_m^2\eta^2u) \geq \frac{(n-2)^2}{4}\|\frac{\mu_m\eta u}{\rho}\|^2 - \|[D, \mu_m\eta]u\|^2 - C_l\|\frac{\mu_m\eta u}{\rho^{1+\frac{\tau}{2}}}\|^2. \tag{7.5}$$

We expand the term $\|[D, \mu_m\eta]u\|^2$ into

$$\begin{aligned}\|[D, \mu_m\eta]u\|^2 &= \|[D, \mu_m]\eta u\| + 2([D, \mu_m]\eta u, \mu_m[D, \eta]u) + \|\mu_m[D, \eta]u\| \\ &\leq \|[D, \mu_m]\eta u\| + C_{\eta,a}\|\chi_{d\eta}u\|^2 \\ &= a^2\|\chi_m\frac{\mu_m\eta u}{\rho}\|^2 + C_{\eta,a}\|\chi_{d\eta}u\|^2.\end{aligned}$$

for m large enough. Here χ_m is the characteristic function of the set $\rho(x) \leq m$, $\chi_{d\eta}$ is the characteristic function of the support of $d\eta$, while $C_{\eta,a} > 0$ is a constant depending on η and a, and independent of u and m. Plugging this into (7.5) we have

$$
\begin{aligned}
&\|\rho^{1+a} D^2 u\| \, \|\eta \frac{\mu_m}{\rho} u\| \geq (\mu_m \rho D^2 u, \frac{\mu_m}{\rho}\eta^2 u) \\
&\geq \left(\frac{(n-2)^2}{4} - a^2\right) \|\chi_m \frac{\mu_m \eta u}{\rho}\|^2 + \frac{(n-2)^2}{4}\|(1-\chi_m)\frac{\mu_m \eta u}{\rho}\|^2 - C_{\eta,a}\|\chi_{d\eta} u\|^2 - C_l \|\frac{\mu_m \eta u}{\rho^{1+\frac{\tau}{2}}}\|^2.
\end{aligned}
$$

Hence for $a \in [0, \frac{n-2}{2})$ fixed, $\|\chi_m \frac{\mu_m \eta u}{\rho}\|$ is uniformly bounded as $m \to \infty$. Therefore, $\rho^{a-1}\eta u \in L^2(M \setminus V, S)$. ☐

In the case when $D^2 u$ vanishes on the asymptotically flat ends, the previous result is simply the integral version of the fact that L^2 spinors u that satisfy $D^2 u = 0$ at infinity, decay pointwise like $\mathcal{O}(\rho^{-n+2})$ on the ends, and thus $\rho^{\frac{n}{2}-2-\epsilon} u \in L^2(M \setminus V, S)$ for all $\epsilon > 0$. This result can be sharpened for spinors that satisfy $Du = 0$ at infinity; such spinors decay like $\mathcal{O}(\rho^{-n+1})$ on the ends, and thus $\rho^{\frac{n}{2}-1-\epsilon} u \in L^2(M \setminus V, S)$. We provide the intergal version of this estimate too.

Proposition 7.6. *Let (M, g) be a nonspin Riemannian manifold which is asymptotically flat of order $\tau > 0$ and has nonnegative scalar curvature. Assume u is a smooth spinor so that Du is compactly supported and $\rho^{-b} u \in L^2(M \setminus V, S)$ for some $b \in [0, \frac{n-2}{2})$. Then,*

$$
\rho^{\frac{n}{2}-1-\epsilon} u \in L^2(M \setminus V, S)
$$

for all $\epsilon > 0$.

Proof. Let η be a radial cutoff function supported in the region of M_l where $\rho(x) \geq L$ and identically 1 in the region where $\rho(x) > 2L$. We choose L large enough so that on the support of η we have $Du = 0$. For a fixed $a > b$ and for each positive integer m, we consider the sequence of functions

$$
\mu_m(\rho) := \begin{cases} \rho^a & \text{if } \rho \leq m, \\ \frac{m^{b+a}}{\rho^b} & \text{if } \rho > m. \end{cases}
$$

Since $Du = 0$ on the support of η, it follows that

$$
D(\mu_m \eta u) = [D, \mu_m \eta] u \in L^2(M \setminus V, S).
$$

Since by construction $\mu_m \eta u \in L^2(M \setminus V, S)$, it follows that $\mu_m \eta u \in \mathrm{Dom}_{\min}(D) \subset H^1_\rho(M \setminus V)$. Therefore we can apply both the Lichnerowicz formula (4.7) and the Poincaré inequality (3.8) to this sequence of spinors.

The Lichnerowicz formula gives

$$
\|[D, \mu_m \eta] u\|^2 \geq \|\nabla(\mu_m \eta u)\|^2. \tag{7.7}
$$

Now we want to make use of the so-far unused orthogonal directions to ν, the unit vector in the direction of $\frac{\partial}{\partial \rho}$, in the term $||\nabla(\mu_m \eta u)||^2$. For this, let ∇^0 denote the covariant derivative in directions orthogonal to ν. The equation $Du = 0$ implies that we have the pointwise inequality

$$
\begin{aligned}
|\mu_m \eta c(\nu) \nabla_\nu u|^2 &= |\mu_m \eta \left(D - c(\nu)\nabla_\nu\right) u|^2 \\
&\leq (n-1)|\mu_m \eta \nabla^0 u|^2.
\end{aligned}
$$

With this, (7.7) becomes

$$\|[D, \mu_m \eta]u\|^2 \geq \|\nabla_\nu(\mu_m \eta u)\|^2 + \frac{1}{n-1}\|\mu_m \eta \nabla_\nu u\|^2 \tag{7.8}$$

For sufficiently large m, the last term on the right-hand-side is greater than $\frac{1}{n-1}\|\chi_m \rho^a \eta \nabla_\nu u\|^2$, with χ_m the characteristic function of the set $\rho \leq m$. For this we have the following weighted Poincaré inequality:

$$\|\chi_m \rho^a \eta \nabla_{e_\rho} u\|^2 \geq \frac{(n+2a-2)^2}{4}\|\frac{\chi_m \rho^a \eta u}{\rho}\|^2 - C_1\|\frac{\chi_m \rho^a \eta u}{\rho^{1+\frac{\tau}{2}}}\|^2.$$

This inequality is proved in the same fashion as the Poincaré inequality in Proposition 3.7. We remark that the boundary term corresponding to $\rho = m$ arising in the bounded domain $1 \leq \rho \leq m$ of this Poincaré inequality may be discarded because its sign is fixed and helps rather than hurts the estimate. The other boundary term does not contribute, since η vanishes there.

Expanding $[D, \mu_m \eta]u$ in (7.8), and using the usual Poincaré inequality for the term $\|\nabla_\nu(\mu_m \eta u)\|^2$ and the above weighted Poincaré inequality for $\frac{1}{n-1}\|\mu_m \eta \nabla_\nu u\|^2$, we obtain

$$\begin{aligned}
&\|\mu_m|d\eta|u\|^2 + 2(|u|\eta d\mu_m, |u|\mu_m d\eta) \\
&\qquad \geq \frac{(n-2)^2}{4}\|\frac{\mu_m \eta u}{\rho}\|^2 + \frac{(n+2a-2)^2}{4(n-1)}\|\frac{\chi_m \mu_m \eta u}{\rho}\|^2 - \|\eta d\mu_m u\|^2 - C_2\|\frac{\mu_m \eta u}{\rho^{1+\frac{\tau}{2}}}\|^2 \\
&\qquad \geq \left(\frac{(n-2)^2}{4} + \frac{(n+2a-2)^2}{4(n-1)} - a^2\right)\|\frac{\chi_m \mu_m \eta u}{\rho}\|^2 \\
&\qquad\qquad + \left(\frac{(n-2)^2}{4} - b^2\right)\|\frac{(1-\chi_m)\mu_m \eta u}{\rho}\|^2 - C_2\|\frac{\mu_m \eta u}{\rho^{1+\frac{\tau}{2}}}\|^2.
\end{aligned}$$

Since $d\eta$ is compactly supported, the left-hand-side is uniformly bounded as $m \to \infty$. Moreover, since $b < \frac{n-2}{2}$, after eventually shrinking the support of η by choosing L larger, the negative term on the right-hand-side can be absorbed into the first two terms, as long as a satisfies

$$\frac{(n-2)^2}{4} + \frac{(n+2a-2)^2}{4(n-1)} - a^2 > 0. \tag{7.9}$$

Equivalently for $n > 2$,

$$\frac{n^2}{4} - \frac{n}{2} > a^2 - a, \tag{7.10}$$

which (for $n > 2$) holds for all $a \in [0, \frac{n}{2})$. Hence taking limit as $m \to \infty$, it follows that

$$\rho^{a-1}u \in L^2(M \setminus V, S) \tag{7.11}$$

as long as $a > \frac{n}{2}$. Thus $\rho^{\frac{n}{2}-1-\epsilon}u \in L^2(M \setminus V, S)$ for all $\epsilon > 0$. □

7.1. More estimates on the asymptotically flat ends of M

As a further application of Agmon's identity (7.2), we use it to derive L^p-estimates for spinors on the asymptotically flat ends of M. These results will not be used for the proof of our main theorems but are useful in the proof of pointwise estimates.

Proposition 7.12. *Let (M, g) be a nonspin Riemannian manifold which is asymptotically flat of order $\tau > 0$ and has nonnegative scalar curvature. Let $u \in H^1_\rho(M \setminus V, S)$ be a smooth spinor so that $\rho^{b+1}|D^2 u|$ is bounded for some $b \geq 1$. Then*

$$\frac{|u|^p}{\rho} \in L^2(M_l) \tag{7.13}$$

for all $p \geq 1$ and for all asymptotically flat ends (M_l, Y_l) of M.

Proof. Fix an asymptotically flat end M_l, and let $0 \leq \eta \leq 1$ be a smooth cutoff function supported on M_l in the region $\rho(x) \geq L$ and equal to 1 in the region $\rho(x) \geq 2L$. We choose a sequence $\{\gamma_j\}$ of smooth cutoff functions, $0 \leq \gamma_j \leq 1$, compactly supported on M, supported in the region $\rho(x) \leq 2j$, identically 1 in the region $\rho(x) \leq j$, and so that $|d\gamma_j| \leq \frac{2}{\rho}$. Let $\eta_j := \eta\gamma_j$.

We consider the bounded positive smooth function $f : (0, +\infty) \to (0, +\infty)$

$$f(t) = \frac{t^p}{1 + at^p}, \tag{7.14}$$

for some positive constant $a > 0$ and $p \geq 0$. Observe that this function satisfies

$$\left(\frac{d}{dt}(tf(t))\right)^2 - \left(t\frac{d}{dt}f(t)\right)^2 \geq \frac{1}{(p+1)^2}\left(\frac{d}{dt}(tf(t))\right)^2. \tag{7.15}$$

Define the sequence of bounded functions on M,

$$f_j(x) := \eta_j(x) f(|u(x)|),$$

to which we apply Agmon's identity (7.2):

$$\begin{aligned}
(D^2u, f_j^2 u) &\geq \|\nabla(f_j u)\|^2 - \|[D, f_j]u\|^2 \\
&\geq \|\eta_j \nabla(f(|u|)u)\|^2 + 2(\eta_j\nabla(f(|u|)u), d\eta_j \otimes f(|u|)u) + \|f(|u|)|u|d\eta_j\|^2 \\
&\quad - \|f(|u|)|u|d\eta_j\|^2 - 2(d\eta_j \otimes f(|u|)u, \eta_j d(f(|u|)) \otimes u) - \|\eta_j|u|d(f(|u|))\|^2 \\
&\geq \|\eta_j d(f(|u|)|u|)\|^2 + 2(\eta_j\nabla(f(|u|)u), d\eta_j \otimes f(|u|)u) - 2(d\eta_j \otimes f(|u|)u, \eta_j d(f(|u|)) \otimes u) \\
&\quad - \|\eta_j|u|d(f(|u|))\|^2,
\end{aligned}$$

where on the last line we have used Kato's inequality (3.13). To estimate the left-hand-side we use the boundness of $\rho^{b+1}|D^2u|$ from the hypothesis, while on the right-hand-side we use the inequality (7.15) to estimate the first and the fourth term, and we group together the second and the third, to get

$$\|\rho^{b+1}D^2u\|_{L^\infty}\ \|\rho^{-b-1}\eta_j^2 f(|u|)^2|u|\|_{L^1} \geq \frac{1}{(p+1)^2}\|\eta_j d(f(|u|)|u|)\|^2 + \frac{1}{2}\langle d\eta_j^2, f(|u|)^2 d|u|^2\rangle.$$

Observe that

$$\eta_j(x)^2 f(|u|(x))^2|u|(x) \leq \begin{cases} |u|(x)^2 & \text{if } |u(x)| \leq 1 \\ a^{-2}|u|(x)^2 & \text{if } |u(x)| \geq 1. \end{cases}$$

Note that since $b \geq 1$, $\rho(x)^{-b-1}|u(x)|^2 \in L^1(M)$. Hence by Lebesgue's dominated convergence theorem, the term $\|\rho^{-b-1}\eta_j^2 f(|u|)^2 u\|_{L^1}$ converges to $\|\rho^{-b-1}\eta^2 f(|u|)^2 u\|_{L^1}$ as $j \to \infty$.

Moreover, for $j \geq 2L$ we have $d\eta_j^2 = d\eta^2 + d\gamma_j^2$, and since all the terms containing $d\gamma_j$ converge to 0 as $j \to \infty$, we can take the limit as $j \to \infty$ to obtain

$$\|\rho^{b+1} D^2 u\|_{L^\infty} \, \|\rho^{-b-1}\eta^2 f(|u|)^2 u\|_{L^1} \geq \frac{1}{(p+1)^2}\|\eta d(f(|u|)|u|)\|^2 + \frac{1}{2}(d\eta^2, f(|u|)^2 d|u|^2).$$

To apply the Poincaré inequality (3.8), we rewrite the above as

$$\begin{aligned}
&\|\rho^{b+1} D^2 u\|_{L^\infty} \, \|\rho^{-b-1}\eta^2 f(|u|)^2 |u|\|_{L^1} \\
&\qquad \geq \frac{1}{(p+1)^2}\|d(\eta f(|u|)|u|)\|^2 - \frac{2}{(p+1)^2}(|u|d(\eta f(|u|)), f(|u|)|u|d\eta) \\
&\qquad\qquad + \frac{1}{(p+1)^2}\|f(|u|)|u|d\eta\|^2 + \frac{1}{2}(d\eta^2, f(|u|)^2 d|u|^2) \\
&\qquad \geq \frac{(n-2)^2}{4(p+1)^2}\|\frac{\eta f(|u|)|u|}{\rho}\|^2 - C_l\|\frac{\eta f(|u|)|u|}{\rho^{1+\tau/2}}\|^2 - \frac{1}{(p+1)^2}\|f(|u|)|u|d\eta\|^2 \\
&\qquad\qquad + \frac{1}{2}(1 - \frac{1}{(p+1)^2})\langle d\eta^2, f(|u|)^2 d|u|^2\rangle - \frac{1}{2(p+1)^2}(d(\eta^2), |u|^2 df^2(|u|)).
\end{aligned}$$

Since $d\eta$ is compactly supported, all the terms containing $d\eta$ are bounded by a constant $C_1 = C_1(p, \|\chi_{d\eta}|u|\|_{L^\infty}, \|\chi_{d\eta} d(|u|)\|_{L^\infty}) > 0$ which is independent of j and of a. Here $\chi_{d\eta}$ denotes the characteristic function of the support of $d\eta$. Thus

$$\|\rho^{b+1} D^2 u\|_{L^\infty} \, \|\rho^{-b-1}\eta^2 f(|u|)^2 u\|_{L^1} + C_1 \geq \frac{(n-2)^2}{4(p+1)^2}\|\frac{\eta f(|u|)u}{\rho}\|^2 - C_l\|\frac{\eta f(|u|)u}{\rho^{1+\tau/2}}\|^2$$

Since $\tau > 0$, we can choose L large enough so that the negative term on the right-hand-side is absorbed into the positive term. Therefore,

$$C_2\|\rho^{-b-1}\eta^2 f(|u|)^2 u\|_{L^1} + C_3 \geq \|\frac{\eta f(|u|)u}{\rho}\|^2$$

for positive constants C_2, C_3 independent of a. Note that since $tf(t)^2 \leq t^{2p+1}$, applying this to the left-hand-side of the above, we obtain

$$C_2\|\frac{\eta |u|^{p+\frac{1}{2}}}{\rho}\|^2 + C_3 \geq \|\frac{\eta f(|u|)u}{\rho}\|^2$$

for all $a > 0$. Assuming that $\frac{\eta|u|^{p+\frac{1}{2}}}{\rho} \in L^2(M_l)$ and taking the limit as $a \to 0$, it follows that $\frac{\eta|u|^{p+1}}{\rho} \in L^2(M_l)$. Now the argument follows by induction, since we know that $\frac{u}{\rho} \in L^2(M \setminus V, S)$. $\square$

8. Coercivity for the Dirac operator

In this section we prove two coercivity results. The first one is for the Dirac operator on $H^1_\rho(M \setminus V, S)$:

Theorem 8.1. *Let (M, g) be a nonspin Riemannian manifold which is asymptotically flat of order $\tau > 0$ and has nonnegative scalar curvature. Let S be the spinor bundle of a maximal spin structure on $M \setminus V$, with V a stratified space given by Theorem 2.7, and D the corresponding Dirac operator. Then, there exists a constant $\lambda > 0$ so that*

$$\|Du\| \geq \lambda \|\frac{u}{\rho}\| \tag{8.2}$$

for all u in $H^1_\rho(M \setminus V, S)$.

As a consequence we derive an invertibility result for the Dirac Laplacian.

Corollary 8.3. *Assume the hypotheses of Theorem 8.1 hold. Then for each smooth spinor Ψ on $M \setminus V$ so that $\rho\Psi \in L^2(M \setminus V, S)$, there exists a unique spinor $\Phi \in H^1_\rho(M \setminus V, S)$ so that $D^2\Phi = \Psi$.*

Proof of Corollary 8.3. Let B be the bilinear form on $H^1_\rho(M \setminus V, S)$ defined by

$$B(u, v) := (Du, Dv).$$

Clearly B is bounded on $H^1_\rho(M \setminus V, S)$. By Equation 8.2 and the Lichnerowicz formula (4.7), we have

$$B(u, u) \geq \frac{1}{2}\|\nabla u\|^2 + \frac{\lambda^2}{2}\|\frac{u}{\rho}\|^2 \geq C\|u\|^2_{H^1_\rho},$$

with $C = \min\{\frac{1}{2}, \frac{\lambda^2}{2}\}$. Thus, B satisfies the conditions of the Lax-Milgram Lemma. We apply this lemma to the linear functional on $H^1_\rho(M \setminus V, S)$,

$$L(v) := (\Psi, v),$$

which is bounded, since $\rho\Psi \in L^2(M \setminus V, S)$. Hence, there exists a unique $\Phi \in H^1_\rho(M \setminus V, S)$ so that

$$B(\Phi, v) = L(v).$$

In particular,

$$(D^2\Phi, v) = (\Psi, v)$$

for all $v \in \mathcal{C}^\infty_0(M \setminus V, S)$. This implies that Φ is a weak solution to $D^2\Phi = \Psi$. Since Ψ is smooth, elliptic regularity implies that Φ is smooth and is thus a strong solution. $\square$

Proof of Theorem 8.1

Because $\mathrm{Dom}_{\min}(D)$ is dense in $H^1_\rho(M \setminus V, S)$, it suffices to prove inequality (8.2) for $u \in \mathrm{Dom}_{\min}(D)$. By Corollary 5.5, the null-space of D on its minimal domain is trivial. We need to show that 0 is not in the essential spectrum of D. We prove this by contradiction.

Let $\{u_j\}$ be an infinite L^2_ρ-orthonormal sequence of sections with $u_j \in \mathrm{Dom}_{\min}(D)$ satisfying $\|Du_j\|_{L^2} \to 0$. Since $R \geq 0$, the Lichnerowicz formula (4.7) implies that $\{u_j\}$ is a bounded sequence in $H^1_\rho(M \setminus V, S)$. By the compactness Lemma 4.3, we may pass to a subsequence (still denoted $\{u_j\}$) which converges strongly on compacta in L^2_ρ and weakly in H^1_ρ to a section $u \in H^1_\rho(M \setminus V, S)$. Since $D : H^1_\rho(M \setminus V, S) \to L^2(M \setminus V, S)$ is bounded, weak H^1_ρ-convergence implies that u lies in the null-space of D. By Corollary 5.5, it follows that $u = 0$. We show that our hypotheses prohibit this and arrive at a contradiction.

Consider the sequence $\{u_j\} \subset \mathrm{Dom}_{\min}(D)$ which converges to zero strongly on compacta in L^2_ρ and weakly in H^1_ρ. Next, we observe that the sequence must also converge to zero in L^2_ρ-norm on the asymptotically flat ends. To see this, choose an end, M_l, and let η be a cutoff function, which is supported in M_l and identically 1 in a neighborhood of infinity in M_l. Clearly $\|D(\eta u_j)\|_{L^2} \to 0$. Moreover the Lichnerowicz formula and Kato's inequality combined with the Poincaré estimate (3.8) on this asymptotically flat end give

$$\|D(\eta u_j)\|^2 \geq \|\nabla(\eta u_j)\|^2 \geq \|d|\eta u_j|\|^2 \geq \frac{(n-2)^2}{4}\|\frac{\eta u_j}{\rho}\|^2 - C_l\|\frac{\eta u_j}{\rho^{1+\tau/2}}\|^2.$$

Shrinking the support of η, we can absorb the negative term above and conclude that $\|\frac{\eta u_j}{\rho}\|_{L^2} \to 0$, as claimed. Since $\rho = 1$ in a neighborhood of V, it follows that the L^2-mass of the sequence accumulates in an arbitrarily small neighborhood of V. We show that this cannot happen.

We first show that for any stratum V^{k_b} of V and any $W \in \mathrm{TRC}(V^{k_b})$, the sequence $\{\|\frac{u_j}{r_b}\|_{L^2(W)}\}$ is bounded. To see this, let $W' \in \mathrm{TRC}(V^{k_b})$ be an open set containing W, and $\zeta \in \mathcal{C}_0^\infty(W')$ with $0 \le \zeta \le 1$ so that $\zeta \equiv 1$ on W. Since

$$\|D(\zeta u_j)\|^2 \le 2\|[D,\zeta]u_j\|^2 + 2\|Du_j\|^2,$$

our hypothesis implies that $\{D(\zeta u_j)\}$ is uniformly bounded in L^2-norm. Since $\zeta u_j \in \mathrm{Dom}_{\min}(D)$, when $k_b > 2$ the estimate (5.4) gives

$$\frac{(k_b-2)^2}{4}\|\frac{\zeta u_j}{r_b}\|^2 \le \|D(\zeta u_j)\|^2 + C_{W'}\|\frac{\zeta u}{r_b^{1/2}}\|^2,$$

while for $k_b = 2$ the estimate (5.2) gives

$$\frac{1}{4}\|\frac{\zeta u_j}{r}\|^2 \le \|D(\zeta u_j)\|^2 + C_{W'}\|\frac{\zeta u}{r^{1/2}}\|^2,$$

with $C_{W'}$ a positive constant depending on W'. Shrinking the radii of the tubular neighborhoods W and W', we can absorb the last terms into the left-hand-side terms of each of the above formulas, and conclude that the sequence $\{\|\frac{\zeta u_j}{r_b}\|_{L^2}\}$ is uniformly bounded.

Since in L^2-norm the sequence $\{u_j\}$ converges to zero on any compact subset of $M \setminus V$, while $\{\|\frac{u_j}{r_b}\|_{L^2(W)}\}$ is uniformly bounded for any $W \in \mathrm{TRC}(V^{k_b})$, it follows that $\{u_j\}$ converges to zero in L^2-norm on any $W \in \mathrm{TRC}(V^{k_b})$. Thus the L^2-mass of the sequence cannot accumulate in arbitrarily small neighborhoods of V, contradicting the above. $\square$

8.1. A second coercivity result

As we will see in the next section, the invertibility result of Corollary 8.3 suffices to prove Theorem A, the existence of Witten spinors, in all the cases except the case when $n = 4$ and $\tau \in (\frac{n-2}{2}, \frac{n}{2}]$. The corollary is insufficient for this exceptional case because the spinor Ψ for which we need to apply Corollary 8.3 is only in $L^2(M \setminus V, S)$ and not in $\rho L^2(M \setminus V, S)$ in this case. To cover the exceptional case, we prove a coercivity result on a weighted Hilbert space with weight shifted from that of $H^1_\rho(M \setminus V, S)$.

Define $\mathcal{H}(M \setminus V, S)$ to be the closure of $\mathcal{C}_0^\infty(M \setminus V, S)$ in the norm

$$\|u\|^2 + \|\rho \nabla u\|^2. \tag{8.4}$$

Note that $\mathcal{H}(M \setminus V, S) \subset \mathrm{Dom}_{\min}(D) \subset H^1_\rho(M \setminus V, S)$, and therefore D has trivial null-space on $\mathcal{H}$.

Theorem 8.5. *Assume that (M, g) satisfies the hypotheses of Theorem 8.1. Then there exists a constant $\lambda > 0$ so that*

$$\|\rho D u\| \ge \lambda \|u\| \tag{8.6}$$

for all u in $\mathcal{H}(M \setminus V, S)$.

Proof. The proof is similar to the proof of Theorem 8.1 except that we need new estimates on the asymptotically flat ends of M, where we have modified the norms of our Hilbert space.

Assume there exists an infinite L^2-orthonormal sequence $\{u_j\}$ in $\mathcal{H}(M \setminus V, S)$ so that $\|\rho D u_j\|_{L^2} \to 0$. The sequence is clearly a bounded sequence in $H^1_\rho(M \setminus V, S)$. The same argument as in Theorem 8.1 then gives that the sequence converges strongly to zero in L^2-norm on compacta in $M \setminus V$ and on any compact neighborhood of V. Therefore the L^2-mass of the sequence must accumulate on the asymptotically flat ends of M.

We show that there exist constants $A > 0$, $L > 0$ so that

$$\|\rho D u\| \geq A\|u\| \quad \text{for all } u \in \mathcal{H}(M \setminus V, S) \text{ with } \operatorname{supp}(u) \subset \{x \in M_l : \rho(x) > L\}, \tag{8.7}$$

where M_l is an asymptotically flat end of M.

Assuming this for the moment, let η be a cutoff function supported on one of the ends. Then $\|\rho D(\eta u_j)\| \to 0$, and then (8.7) shows that the L^2-mass of the sequence $\{u_j\}$ cannot accumulate on the asymptotically flat ends either. Thus we reach a contradiction.

It remains to show (8.7). Since $\mathcal{C}^\infty_0(M \setminus V, S)$ is dense in $\mathcal{H}(M \setminus V, S)$, it suffices to prove (8.7) for $u \in \mathcal{C}^\infty_0(M \setminus V, S)$. We write

$$\|\rho D u\|^2 = \|D(\rho u)\|^2 - 2(D(\rho u), [D, \rho]u) + \|[D, \rho]u\|^2.$$

Let ν denote the unit vector in the direction of $\frac{\partial}{\partial \rho}$, and let ∇^0 denote the covariant derivative in directions $\{e_\sigma\}_{\sigma=2,\dots,n}$ orthogonal to ν. We apply the Lichnerowicz formula to the first term on the right-hand-side and expand the cross-term, to

$$\begin{aligned}\|\rho D u\|^2 \geq \|\nabla_\nu(\rho u)\|^2 + \|\nabla^0(\rho u)\|^2 - 2(c(\nu)\nabla_\nu(\rho u), [D, \rho]u) \\ - 2((D - c(\nu)\nabla_\nu)(\rho u), [D, \rho]u) + \|[D, \rho]u\|^2.\end{aligned}$$

Observe that we can group

$$\|\nabla^0(\rho u)\|^2 - 2((D - c(\nu)\nabla_\nu)(\rho u), [D, \rho]u) + (n-1)\|[D, \rho]u\|^2 = \sum_{\sigma=2}^n \|\nabla_{e_\sigma}(\rho u) + c(e_\sigma)[D, \rho]u\|^2,$$

and obtain

$$\begin{aligned}\|\rho D u\|^2 \geq \|\nabla_\nu(\rho u)\|^2 + \sum_{\sigma=2}^n \|\nabla_{e_\sigma}(\rho u) + c(e_\sigma)[D, \rho]u\|^2 \\ - 2(c(\nu)\nabla_\nu(\rho u), [D, \rho]u) - (n-2)\|[D, \rho]u\|^2\end{aligned}$$

Since the metric is asymptotically flat of order τ, $|[D, \rho]| = 1 + \mathcal{O}(\rho^{-\tau})$, and thus

$$\|\rho D u\|^2 \geq \|\nabla_\nu(\rho u)\|^2 - 2(\rho \nabla_\nu u, u) - n\|u\|^2 - C_1 \|\frac{u}{\rho^{\tau/2}}\|^2,$$

for some constant $C_1 > 0$ independent of u. To handle the term $-2\langle \rho \nabla_\nu u, u\rangle$, we integrate by parts to rewrite it as

$$\begin{aligned}-(\nabla_\nu |u|^2, \rho) &= (|u^2|, \rho^{1-n}\nabla_\nu(\rho^n)) - C_2 \|\frac{u}{\rho^{\tau/2}}\|^2 \\ &= n\|u\|^2 - C_3 \|\frac{u}{\rho^{\tau/2}}\|^2.\end{aligned}$$

The error terms arise from the deviation of the metric from the Euclidean metric. Thus, we obtain

$$\|\rho Du\|^2 \geq \|\nabla_\nu(\rho u)\|^2 - C\|\frac{u}{\rho^{\tau/2}}\|.$$

with $C > 0$ a constant independent of u. Now the desired inequality follows using the Poincaré inequality (3.8) on the asymptotically flat end and choosing L sufficiently large so that the lower order term can be absorbed into $\frac{(n-2)^2}{4}\|u\|^2$. □

As a consequence, we have the following invertibility result, analogous to Corollary 8.3.

Corollary 8.8. *Assume that the hypotheses of Theorem 8.5 hold. Then for each smooth spinor $\Psi \in L^2(M\backslash V, S)$, there exists a unique spinor Φ in $\mathcal{H}(M\backslash V, S)$ so that $D(\rho^2 D\Phi) = \Psi$.*

Proof. The only difference from the proof of Corollary 8.3 is that we now take B to be the bilinear form on $\mathcal{H}(M \setminus V, S)$ defined as

$$B(u, v) := (\rho Du, \rho Dv),$$

and apply the Lax-Milgram Lemma to the bounded linear functional L on $\mathcal{H}(M \setminus V, S)$ defined to be $L(v) = (\Psi, v)$.

The only argument which requires a slightly different justification is showing that the bilinear form B is coercive. For this, let $\epsilon > 0$ small to be chosen later, and bound

$$\|\rho Du\|^2 \geq \epsilon\|\rho Du\|^2 + (1-\epsilon)\lambda^2\|u\|^2 \tag{8.9}$$

using (8.6). To estimate the first term on the right-hand-side, note that since $u \in \mathcal{H}(M \setminus V, S)$, it follows that $\rho u \in H^1_\rho(M \setminus V, S)$. Thus using the Lichnerowicz formula (4.7), we have

$$\begin{aligned}
\|\rho Du\|^2 &= \|D(\rho u)\|^2 + \||d\rho|u\|^2 - 2(D(\rho u), c(d\rho)u) \\
&\geq \frac{1}{2}\|D(\rho u)\|^2 - \||d\rho|u\|^2 \\
&\geq \frac{1}{2}\|\nabla(\rho u)\|^2 - \||d\rho|u\|^2 \\
&\geq \frac{1}{2}\|\rho\nabla u\|^2 + \frac{1}{2}\||d\rho|u\|^2 + (\rho\nabla u, d\rho \otimes u) - \||d\rho|u\|^2 \\
&\geq \frac{1}{4}\|\rho\nabla u\|^2 - \frac{3}{2}\||d\rho|u\|^2
\end{aligned}$$

Choosing ϵ so that $\frac{3\epsilon}{2}\|d\rho\|^2_{L^\infty(M)} < \frac{1}{2}(1-\epsilon)\lambda^2$ gives the coercivity of the bilinear form B on the Hilbert space $\mathcal{H}(M \setminus V, S)$. □

9. Proof of our main results

In this section we prove our main theorems stated in the Introduction. For the proof of Theorem A, the existence and construction of the Witten spinor is separated into two cases depending on the order of convergence, τ, of the asymptotically flat metric to a Euclidean metric. The reason for this is that, for $\tau > \frac{n}{2}$, a spinor ψ_0 supported on an end and constant in a frame induced from an asymptotically flat coordinate system satisfies $\rho D\psi_0 \in L^2(M \setminus$

$V, S)$; the existence of the Witten spinor is then an immediate consequence of Corollary 8.3. However, if $\tau \in (-\frac{n-2}{2}, \frac{n}{2}]$, then $\rho D\psi_0$ need not be L^2, but $\rho D^2\psi_0$ is still square integrable. Establishing the existence of the Witten spinor from this weaker hypothesis is a two step procedure, provided that $n \geq 5$. In the case $n = 4$ and $\tau \in (-\frac{n-2}{2}, \frac{n}{2}]$ the proof requires further refinement.

The proofs of Theorem B and Theorem C are based on the form of the Witten spinor derived in Theorem A. As a consequence, we separate these proofs into cases, according to the construction we use for the Witten spinor.

9.1. Proof of Theorem A

Let ψ_0 be a smooth spinor which is constant on the asymptotically flat ends of M and suported outside a neighborhood of V. It follows (see (4.13)) that $\rho^{\tau+1}|D\psi_0|$ is bounded on $M \setminus V$. We separate the construction into two cases, according to whether $\frac{n-2}{2} < \tau \leq \frac{n}{2}$ or $\tau > \frac{n}{2}$.

If $\tau > \frac{n}{2}$, then $\rho D\psi_0 \in L^2(M \setminus V, S)$, and thus the spinor $D\psi_0$ satisfies the hypothesis of Corollary 8.3. Hence, there exists a unique $u \in H^1_\rho(M \setminus V, S)$ so that

$$D^2 u = -D\psi_0.$$

From Corollary 4.14, it follows that the spinor $\psi := Du + \psi_0$ is a Witten spinor.

If $\tau \in (\frac{n-2}{2}, \frac{n}{2}]$, more work is required to construct the desired Witten spinor. In this case, $D^2\psi_0$ satisfies the hypothesis of Corollary 8.3. Hence there exists a unique $w \in H^1_\rho(M \setminus V, S)$ so that

$$D^2 w = -D^2\psi_0.$$

Let

$$W := w + \psi_0.$$

Then $D^2 W = 0$, $DW \in L^2(M \setminus V, S)$, and DW is in the null-space of the maximal extension of the Dirac operator. If in fact $DW = 0$, then W is the desired Witten spinor. If $DW \neq 0$, then we modify W further. Let η be a smooth cutoff function, vanishing in a neighborhood of V and identically 1 outside of a compact neighborhood of V. Without loss of generality, we can assume that η is 1 on the support of ψ_0. Since $DW \in L^2(M \setminus V, S)$ is strongly harmonic, it follows that $\eta DW \in H^1_\rho(M \setminus V, S)$ and $D(\eta DW)$ is compactly supported. By Proposition 7.6, $\rho^{\frac{n}{2}-1-\epsilon}(\eta DW) \in L^2(M \setminus V, S)$ for all $\epsilon > 0$. In particular, when $n \geq 5$ we have $\rho D(\eta W) \in L^2(M \setminus V, S)$ and by Corollary 8.3 there exists a unique $u \in H^1_\rho(M \setminus V, S)$ so that

$$D^2 u = -D(\eta W).$$

Then, as in the previous case, Corollary 4.14 gives that the spinor $\psi := Du + \eta W = Du + \eta w + \psi_0$ is a Witten spinor.

We are left to analyze the case $\tau \in (-\frac{n-2}{2}, \frac{n}{2}]$ and $n = 4$. In this case, we use our second coercivity result in Section 8.1 to construct the Witten spinor. Since $D\psi_0 \in L^2(M \setminus V, S)$, by Corollary 8.8 there exists a unique $u \in L^2(M \setminus V, S)$ with $\rho\nabla u \in L^2(M \setminus V, S)$ so that

$$D(\rho^2 Du) = -D\psi_0. \tag{9.1}$$

We set $v := \rho^2 Du$ and let $\psi := v + \psi_0$. Since $\rho Du \in L^2(M \setminus V, S)$, then $\frac{v}{\rho} \in L^2(M \setminus V, S)$. Moreover $Dv = -D\psi_0 \in L^2(M \setminus V, S)$ and then Proposition 4.11 gives $\nabla v \in L^2(M_l, S|_{S_l})$ for all asymptotically flat ends (M_l, Y_l) of M. Hence ψ is a Witten spinor. $\square$

Remark 9.2. Note that the spinor $\psi - \psi_0$ satisfies the hypothesis of Proposition 7.3 with $b = -1$. Thus $\rho^{\frac{n}{2}-1-\epsilon}(\psi - \psi_0) \in L^2(M \setminus V, S)$ for all $\epsilon > 0$.

Remark 9.3. Note that if we could construct our Witten spinor to be $W = w + \psi_0$ with $w \in H^1_\rho(M \setminus V, S)$ solution to $D^2 w = -D^2\psi_0$, it follows that W is in the minimal domain of D near V, and thus by Proposition 4.15 the positive mass theorem holds true.

9.2. Proof of Theorem B

The proof of this theorem is a consequence of the estimates we derived in Section 6.

Consider the Witten spinor ψ given by the proof of Theorem A. Thus $\psi = Du + \psi_0$ in the case when $\tau > \frac{n}{2}$, $\psi = Du + \eta w + \psi_0$ in the case $\tau \in (\frac{n-2}{2}, \frac{n}{2}]$ and $n \geq 5$; while $\psi = \rho^2 Du + \psi_0$ in the case $\tau \in (\frac{n-2}{2}, \frac{n}{2}]$ and $n = 4$. Both ψ_0 and ηw vanish in a neighborhood of V (where $\rho = 1$). Hence $\psi = Du$ in this neighborhood, and therefore also $D^2 u = 0$ there. Moreover, by construction we have have $\chi u \in \mathrm{Dom}_{\min}(D)$ for all $\chi \in \mathcal{C}^\infty_0(M)$. Thus after multiplying by a cutoff function supported in the region where $D^2 u = 0$ and which is identically 1 in a smaller neighborhood of V, u satisfies the hypothesis of Lemma 6.3 and Lemma 6.8. Then Lemma 6.6 gives the desired estimate for ψ near V^2, while Lemma 6.12 gives the estimates near the higher codimension strata V^{k_b}. $\square$

9.3. Proof of Theorem C

The main ingredient for this proof is Proposition 6.14.

Let ψ be the Witten spinor constructed in Theorem A and which satisfies (1.10). Thus $\psi = Du + \psi_0$ in the case when $\tau > \frac{n}{2}$, $\psi = Du + \eta w + \psi_0$ in the case $\tau \in (\frac{n-2}{2}, \frac{n}{2}]$, while $\psi = \rho^2 Du + \psi_0$ in the case $\tau \in (\frac{n-2}{2}, \frac{n}{2}]$ and $n = 4$. Let χ be a smooth cutoff function on M which is supported in a neighborhood of V where $\psi_0 = 0$ and is identically 1 in a smaller neighborhood of V. We show that $\chi\psi \in \mathrm{Dom}_{\min}(D)$. For this we use Proposition 6.14.

Since χ is supported where $\psi_0 = 0$, $\chi\psi = \chi Du$, with $\chi u \in \mathrm{Dom}_{\min}(D)$ and $\chi D^2 u = 0$. Since $\chi\psi = D(\chi u) - [D, \chi]u$, it is enough to show that $D(\chi u) \in \mathrm{Dom}_{\min}(D)$. This follows as a consequence of Proposition 6.14 applied to $\bar{u} := \chi u$ and $\bar{v} := D\bar{u}$, since $\bar{u} \in \mathrm{Dom}_{\min}(D)$, $D^2\bar{u} = 0$ in a neighborhood of V^2, and $\frac{\bar{v}}{r^{1/2}\ln^{1/2}(\frac{1}{r})} \in L^2(W)$ for all $W \in \mathrm{TRC}(V^2)$ by assumption (1.10).

Therefore the spinor ψ satisfies the conditions of Proposition 4.15. Since the scalar curvature is nonnegative, the positivity of the mass follows from formula (4.16). $\square$

References

[ADM] R. Arnowitt, S. Desser and C. Misner, *Coordinate invariance and energy expressions in General Relativity*, Phys. Rev. **122** (1961), 997–1006.

[Ag] S. Agmon, *Lectures on exponential decay of solutions of second order elliptic equations*, Princeton University Press, Princeton, 1982.

[Bal] P. Baldwin, *L^2 solutions of Dirac equations*, PhD Thesis, University of Cambridge, 1999.

[Bar] R. Bartnik, *The mass of an asymptotically flat manifold*, Comm. Pure Appl. Math. **39** (1986), no. 5, 661–693.

[Br] H. Bray, *Proof of the Riemannian Penrose inequality using the positive mass theorem*, J. Diff. Geom. **59** (2001), no. 2, 177–267.

[Gr] A. Gray, *Tubes*, Second edition, Progress in Mathematics, 221, Birkhäuser Verlag, Basel, 2004.

[GL] M. Gromov and B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. No., **58** (1983), 83-196 (1984).

[LM] B. Lawson and M-L. Michelsohn, *Spin geometry*, Princeton University Press, Princeton, 1989.

[LP] J. Lee and T. Parker, *The Yamabe problem*, Bull. Amer. Math. Soc. (N.S.), **17** (1987), no. 1, 37-91.

[Lo] J. Lohkamp, *The higher dimensional positive mass theorem I*, preprint, arXiv:math/0608795v1.

[MS] J. Milnor and J. Stasheff, *Characteristic Classes*, Princeton University Press, Princeton, 1974.

[PS1] W. Pardon and M. Stern, *L^2-$\bar{\partial}$-cohomology of complex projective varieties*, JAMS **4**, (1991), 603–621.

[PS2] W. Pardon and M. Stern, *Pure Hodge Structure on the L^2-cohomology of varieties with isolated singularities*, J. Reine Angew. Math. **533**, (2001), 55–80.

[PT] T. Parker and C. Taubes, *On Witten's proof of the positive energy theorem*, Comm. Math. Phys. **84** (1982), no. 2, 223–238.

[Sc1] R. Schoen, personal communication.

[Sc2] R. Schoen, *On the proof of the positive mass conjecture in general relativity*, J. Diff. Geom. **20** (1984), no. 2, 479–495.

[SY] R. Schoen and S. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76.

[Wi] E. Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), no. 3, 381–402.